Qualifying Exam in Topology, Spring 2019

1. Definitions and Theorems

Define the following terms/state the following theorems. Definitions/Theorems must be stated in full. *In addition, provide briefly an example for each of the definitions (1 – 3).*

1. Hausdorff
2. Complete metric space
3. Covering map $p : X \to Y$
4. Urysohn Metrization Theorem
5. $\pi_1(X, x)$, i.e. the fundamental group of a topological space X with basepoint x. (The definition should include the definition of the group operation. Just state definitions, no verification of well-definedness required.)

2. Point Set Topology (Mostly)

Solve 4 of the following problems. In your answers, indicate what theorems that you are using.

(1) Suppose X is compact and Y is Hausdorff. Show that if $f : X \to Y$ is a surjective continuous map, then f is a quotient map.

(2) For each of the following, prove it's true or prove it's false with a counterexample.

(a) If $B(x, r)$ is an open ball of radius $r > 0$ with center x in a metric space X, then its closure $\overline{B(x, r)}$ is $\{y \in X \mid d(x, y) \leq r\}$.

(b) There is a continuous surjective map $f : [-1, 1] \to [-1, 1]^9$.

(c) If $A \subseteq X$ is connected, then the closure $\overline{A}$ is connected.

(3) (a) State the Baire Category Theorem.

(b) Prove that a complete metric space with no isolated points is uncountable.

(4) Let X be a Hausdorff space. Prove from the definitions that if $Y \subset X$ is compact and $x \in X \setminus Y$, then there are open sets U containing x and V containing Y such that $U \cap V = \emptyset$.

(5) Let $X = \prod_{i \in \mathbb{N}} X_i$ where $X_i = S^3$. Give X the product topology. Show that X is metrizable.

3. Fundamental Group and Covering Spaces (Mostly)

Solve 4 of the following problems. In your answers, indicate what theorems you are using.

(1) Let (X,A) be a pair of topological spaces.

(a) State the homotopy extension property (HEP) for a pair (X,A) of spaces.

(b) Prove the HEP is equivalent to the existence of a retraction $r : X \times I \to (A \times I) \cup (X \times 0)$.

(2) Prove the Brouwer Fixed Point Theorem (in dimension 2).

(3) Let X be a path-connected, locally path-connected, semilocally simply-connected space. Let $p : \tilde{X} \to X$ be the universal covering space of X, and let $A \subseteq X$ be a path-connected, locally path-connected, semilocally simply-connected subspace. Let $\tilde{A}$ be a path-component of $p^{-1}(A)$. Prove that the following are equivalent.

(a) The restriction $p|_{\tilde{A}} : \tilde{A} \to A$ is the universal cover of A.

(b) The map $\pi_1(A, a) \to \pi_1(X, a)$ induced by inclusion is injective.

(4) Let $T^2 = S^1 \times S^1$ be the 2-torus. Show that the set of isomorphism classes of covering maps $T^2 \to T^2$ is countably infinite. (In other words, there are countably many covering maps $T^2 \to T^2$ which are distinct even up to isomorphism of covering maps.)

(5) Let D be the closed unit disc $\{z \in \mathbb{C} \mid |z|^2 \leq 1\} \subset \mathbb{C}$ and let $T^2 = S^1 \times S^1$. Let X be the topological space obtained by gluing ∂D to one factor of T^2 by a 5-fold covering map. More formally, $X = D \sqcup T^2 / \sim$ where the equivalence relation $\sim$ is generated by: $e^t \in \partial D$ is equivalent to $(e^{5t}, 0) \in T^2$. (Otherwise, points are not equivalent.) Compute $\pi_1(X, x)$.