Be sure that this examination has 2 pages

The University of Oklahoma
Qualifying Examination – January 15, 2020

Real Analysis

Closed book examination | Time: 180 minutes

Choose 7 out of the following 8 questions

1. Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of measurable subsets of $\mathbb{R}$. For each $n \in \mathbb{N}$, define B_n to be the set of all points in $\mathbb{R}$ which are contained in at least n of the sets $\{A_k\}_{k=1}^{\infty}$. Prove that B_n is measurable and
$$m(B_n) \leq \frac{1}{n} \sum_{k=1}^{\infty} m(A_k).$$

2. Find two disjoint subsets $E_1, E_2 \subset \mathbb{R}$ such that
$$m_*(E_1 \cup E_2) < m_*(E_1) + m_*(E_2).$$

3. Assume that $f_n \in L^1(E)$ for some measurable set $E \subseteq \mathbb{R}$. If
$$\sum_{n=1}^{\infty} \int_E |f_n| < \infty,$$
Prove that $f = \sum_{n=1}^{\infty} f_n$ is integrable and
$$\int_E f = \sum_{n=1}^{\infty} \int_E f_n.$$

4. Let $E \subseteq \mathbb{R}$ be a measurable set, and let $f : E \to \mathbb{R}$ be an integrable function. Show that if $\{A_k\}_{k=1}^{\infty}$ is a sequence of measurable subsets of E such that $\lim_{k\to\infty} m(A_k) = 0$, then $\lim_{k\to\infty} \int_{A_k} f = 0$.

5. Show that if $\|f_n - f\|_{L^1} \to 0$, then there is a subsequence of $\{f_n\}_{n=1}^{\infty}$ which converges to f a.e.

6. Let ℓ^1 be the space of all real-valued sequences $x = (x_1, x_2, x_3, \ldots)$ such that $\sum_{j=1}^{\infty} |x_j| < \infty$, with norm $\|x\| = \sum_{j=1}^{\infty} |x_j|$.
   (a) Show that ℓ^1 is complete.
   (b) Give an example of a bounded sequence of elements of ℓ^1, which does not contain any subsequence which converges to a point in ℓ^1.

7. Suppose that $1 \leq p < q < \infty$ and $f \in L^p(\mathbb{R}) \cap L^q(\mathbb{R})$. Show that $f \in L^r(\mathbb{R})$ for each $r \in (p, q)$.

8. Let $f_n, g_n, f, g \in L^2(\mathbb{R})$ be such that $\{f_n\}_{n=1}^{\infty}$ converges strongly to f, and $\{g_n\}_{n=1}^{\infty}$ converges weakly to g. Show that $\lim_{n\to\infty} \int_{\mathbb{R}} f_n g_n = \int_{\mathbb{R}} fg$.

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