Qualifying exam Analysis August 2020

1. **(3+6+3 points)** (a) Let $\mathcal{A}_0$ be the collection of sets $A \subseteq [c,d)$ that are finite disjoint unions of intervals of the form $I = [a,b)$, with $c \le a \le b \le d$. Show that $\mathcal{A}_0$ is an algebra on $X = [c,d)$. (A somewhat informal argument, perhaps supported by a sketch, would be fine here.)

   (b) Let μ, ν be measures on the Borel σ-algebra $\mathcal{B}$ of $X = \mathbb{R}$. Suppose that: (i) $\mu((a,b)) = \nu((a,b))$ and (ii) $\mu((a,b)), \nu((a,b)) < \infty$ for all $a, b \in \mathbb{R}$, $a < b$. Show that then $\mu(B) = \nu(B)$ for all $B \in \mathcal{B}$. *Suggestion:* Use the monotone class theorem (= Theorem 2.10 in Bass's book).

   (c) Show that the conclusion of part (b) can fail if assumption (ii) is dropped.

2. **(4 points)** Let $\mathcal{M} = \{(a, a+1) : a \in \mathbb{R}\}$. Show that $\mathcal{M}$ generates the Borel σ-algebra on $\mathbb{R}$.

3. **(4 points)** Let $f \in L^1(\mathbb{R})$. Show that then

$$g(x) = \int_{-\infty}^{\infty} f(t) \cos(xt^2)\, dt$$

   is a continuous function on $\mathbb{R}$.

4. **(4 points)** Let $\nu = m_\alpha$ be the Borel measure on $\mathbb{R}$ that is generated by the increasing function

$$\alpha(x) = \begin{cases} x & x < 0 \\ x+1 & x \ge 0 \end{cases},$$

   and let μ be the Lebesgue measure. Find the Lebesgue decomposition of ν with respect to μ, that is, find measures λ, ρ such that $\nu = \lambda + \rho$, $\lambda \perp \mu$, $\rho \ll \mu$.

5. **(2+3+2+3 points)** Consider the sequence of functions $f_n \in L^1(\mathbb{R})$, $f_n(x) = n\chi_{(0,1/n)}(x)$. Does f_n converge:
   (a) pointwise almost everywhere (with respect to Lebesgue measure);
   (b) in L^1;

(c) in measure;
(d) in $\mathcal{D}'(\mathbb{R})$?
In those cases where the sequence does converge, please also identify the limit.

6. **(6 points)** Evaluate

$$\int_0^\infty dx \int_1^\infty dy\, e^{(i-1)xy^2}.$$

You will probably want to use Fubini-Tonelli here. Please justify this carefully; don't just do the formal calculation.

7. **(3+4 points)** Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $f \in L^p(\mathbb{R})$ if and only if: (a) $1 < p < 2$; (b) $p \geq 2$

8. **(4 points)** Let $f : (0,1) \to \mathbb{C}$ be a measurable function. Show that $M : [1,\infty) \to [0,\infty]$,

$$M(p) = \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p},$$

is an increasing function of p.

9. **(5 points)** Let $u \in \mathcal{D}'(\mathbb{R})$ and $f \in C^\infty(\mathbb{R})$. Prove the product rule for the distributional derivative:

$$(fu)' = f'u + fu'$$