Algebra Qualifying Exam – January 2021

- *Please try to explain your work clearly and write neatly.*
- *Attempt at least one question from each section. Within this constraint, full credit for complete answers to 6 questions.*
- *All questions carry equal weight and points are split evenly between the parts of a question. For example, # 2 has two parts, so each is worth 50% of the points.*
- *By convention, all rings have a 1. For R a ring, $R^\times$ denotes the group of units of R.*
- *Good luck!*

Section 1 – Groups.

1. For each statement, provide a proof if true, a counterexample if false.

   a. Every group of order 12 is abelian.

   b. Every infinite group contains an element of infinite order.

   c. If a simple group G acts non-trivially on a set with 4 elements then G is cyclic.

2. Let G be a finite group and p be a prime.

   a. Show that G cannot have exactly 2 Sylow p-subgroups. Give an example of a group that has exactly 3 Sylow 2-subgroups.

   b. Let N be a normal subgroup of G and suppose that S is a normal Sylow p-subgroup of N. Prove that S is normal in G.

Section 2 – Rings.

3. Let R be a ring in which $r^2 = r$ for all $r \in R$.

   a. Prove that $r = -r$ for all $r \in R$.

   b. Prove that R is commutative.

   c. Let $\mathfrak{p}$ be a prime ideal in R. (By definition, $\mathfrak{p} \neq R$.) Show that $R/\mathfrak{p} \simeq \mathbb{F}_2$, the field with 2 elements.

4. a. Show that the polynomial ring $\mathbb{Z}[X]$ is not a PID (principal ideal domain).

b. Consider the ring $C[0,1]$ of continuous functions $f : [0,1] \to \mathbb{R}$ under pointwise addition and multiplication. Show that $C[0,1]$ is not Noetherian.

Section 3 – Fields.

5. a. Let K/F be a field extension with $[K : F]$ odd. Show that $F(\alpha) = F(\alpha^2)$ for all $\alpha \in K$.

   b. Is $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$ a Galois extension? Justify your answer.

6. Consider the finite field $\mathbb{F}_{2^6}$ with 2^6 elements.

   a. Describe the structure of the group $\mathrm{Aut}(\mathbb{F}_{2^6})$ of field automorphisms of $\mathbb{F}_{2^6}$.

   b. Describe the lattice of subfields of $\mathbb{F}_{2^6}$. That is, list the subfields of $\mathbb{F}_{2^6}$ and the containments between them—even better, draw a picture!

   c. How many elements $\alpha \in \mathbb{F}_{2^6}$ satisfy $\mathbb{F}_2(\alpha) = \mathbb{F}_{2^6}$? (Hint: α has the given property if and only if α is not contained in a maximal subfield of $\mathbb{F}_{2^6}$.)

Section 4 – Rings and Modules.

7. a. State a version of the Chinese Remainder Theorem.

   b. Consider the ring $R = \mathbb{Q}[\mathrm{X}] \,/\, \big(\mathrm{X}^2 - 1\big)$. Describe the maximal ideals in R.

   c. With R as in part b, write $R^{\times}_{\mathrm{tor}}$ for the torsion subgroup of $R^{\times}$ (made up of the elements of finite order in $R^{\times}$). Determine the group $R^{\times}_{\mathrm{tor}}$.

8. a. Prove that the additive group $\mathbb{Q}$ of rational numbers is not a free $\mathbb{Z}$-module.

   b. Let $\mathrm{M}_3(F)$ denote the ring of 3×3 matrices with entries in a field F. Suppose $A \in \mathrm{M}_3(F)$ satisfies $A^3 = 0$ but $A^2 \neq 0$. Write $\mathcal{C}(A)$ for the centralizer of A, that is,

$$\mathcal{C}(A) = \{B \in \mathrm{M}_3(F) : AB = BA\}.$$

   Determine the dimension of $\mathcal{C}(A)$ as an F-vector space.
