Qualifying Exam in Topology, Spring 2021

1. Concepts

Answer all the questions in this section.

1. Prove or give a counter-example: If X is a topological space and $f : X \to \mathbb{R}$ is a continuous function, then the closure of the subset $\{x \in X \mid f(x) > 0\}$ equals $\{x \in X \mid f(x) \geq 0\}$.

2. Let $\mathcal{B}$ be the collection of subsets $\{[a, b) \mid a, b \in \mathbb{R}, a < b\}$ in $\mathbb{R}$. Prove that $\mathcal{B}$ is a basis for a topology. Prove or disprove that $\mathbb{R}$ is connected in this topology.

3. Prove or disprove that the map $f : [0, 1] \to S^1$ defined by $f(t) = e^{2\pi i t}$ is a covering map.

4. Suppose a subspace $X \subset \mathbb{R}^n$ has a point $x \in X$ with the following property: for every $y \in X$, the line segment from y to x lies in X. Prove, directly from the definition of π_1, that $\pi_1(X, x) = 1$, i.e. the fundamental group is trivial.

2. Point Set Topology (Mostly)

Solve 3 of the following problems. In your answers, indicate which theorems you are using. If you are using a theorem you found in the book that you are using for the exam, give the theorem number or the section it's in plus the page number. If you are using a theorem/result that you remember from class, then describe what the theorem/result says.

(1) Prove that there is a continuous surjective map $[0, 1] \to S^2$. Prove that any such map is a quotient map. (Yes, that is really $[0, 1]$ and not $[0, 1]^2$.)

(2) Suppose $\{X_i\}_{i \in I}$ is a collection of topological spaces. Suppose we choose a subspace $A_i \subseteq X_i$ for all $i \in I$. Let $Y = \prod_{i \in I} X_i$ with the product topology.

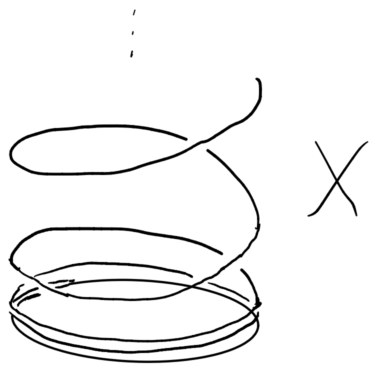
(i) Prove that $\prod_{i \in I} A_i$ is a closed subset of Y if A_i is closed in X_i for all i.

(ii) Prove that $\overline{\prod_{i \in I} A_i} = \prod_{i \in I} \overline{A_i}$.

(3) Let $X \subseteq \mathbb{R}^3$ be the subspace

$$X = S^1 \times \{0\} \cup \{(\cos t, \sin t, e^t) \mid t \in \mathbb{R}\}$$

(See the figure below.) Prove that X is connected but not path-connected.



(4) Prove that the product $\prod_{n\in\mathbb{N}} S^n$ is metrizable (where $\prod_{n\in\mathbb{N}} S^n$ has the product topology).

3. Fundamental Group and Covering Spaces (Mostly)

Solve 3 of the following problems. In your answers, indicate which theorems you are using. If you are using a theorem you found in the book that you are using for the exam, give the theorem number or the section it's in plus the page number. If you are using a theorem/result that you remember from class, then describe what the theorem/result says.

(1) Use the fundamental group (of some space) to prove that S^2 and S^3 are not homeomorphic.

(2) (i) Among the spaces $S^2, \mathbb{R}\mathrm{P}^2, T^2$, determine which can cover the other spaces. Justify your answer. (E.g. is there a covering map $S^2 \to \mathbb{R}\mathrm{P}^2$? $S^2 \to T^2$? etc. Here T^2 denotes the 2-dimensional torus. You can state and use without proof what π_1 of these spaces are.

(ii) Prove that every map $\mathbb{R}\mathrm{P}^2 \to T^2$ is homotopic to a constant map.

(3) Let $X = S^1 \vee S^1 \vee S^2$.

(i) What are all the possible deck transformation groups (up to isomorphism) for a regular index 5 cover of X? Justify your answer.

(ii) Describe two non-isomorphic regular index 5 covers of X and how the deck transformation group acts on those covers.

(4) Let X be the topological space obtained by attaching two distinct tori at two of their respective points as indicated in the diagram. More precisely let

$$X = T^2 \times \{0\} \cup T^2 \times \{1\} / \sim$$

where $(x_1, 0) \sim (x_1, 1)$ and $(x_2, 0) \sim (x_2, 1)$ for distinct points x_1, x_2 in the 2-dimensional torus. Use Van Kampen's Theorem to determine $\pi_1(X, x)$. (Hint: Be careful of the subsets you use for Van Kampen.)