Real Analysis Qualifying Exam – August 2021

NOTATION: $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real and natural numbers, respectively. Unless otherwise specified, the standard metrics/topologies/measures are always assumed for all spaces involved. m denotes Lebesgue measure on the real line. As customary, the integral of f with respect to Lebesgue measure may be written as $\int f(x)dx$ instead of $\int f dm$. $L^p(X,\mathscr{M},\mu)$ denotes the space of $\mathscr{M}$-measurable functions f such that $|f|^p$ is μ-integrable, and the norm of f in this space is denoted $\|f\|_p$ (with the necessary modification in the case $p=\infty$).

There are 8 equally-weighted problems in this exam. Answer as many of them as you can.

(1) Let A be the set of real numbers in $[0,1]$ whose decimal expansions contain no threes. Prove that A is Lebesgue measurable, and find its measure. Some real numbers have non-unique decimal expansions, why does this not cause an issue?

(2) Let $A \subseteq \mathbb{R}$ be a Lebesgue measurable set. Suppose that for any $a,b \in \mathbb{R}$ with $a<b$ we have

$$m\big(A \cap (a,b)\big) \leq \frac{b-a}{2}.$$

Prove that $m(A)=0$.

(3) For each $n \in \mathbb{N}$, let

$$f_n(x) = \frac{(1-x)^n \cos\left(\frac{n}{x}\right)}{\sqrt{x}}.$$

Show that $\lim_{n\to\infty} \int_0^1 f(x)dx$ exists and find its value.

(4) Let $(X,\mathscr{A},\mu)$ be a finite measure space. Suppose $A_n \in \mathscr{A}$ for each n, and the indicator functions χ_{A_n} converge in $L^1(X,\mathscr{A},\mu)$ to a function f. Prove that there exists $A \in \mathscr{A}$ such that f and χ_A are equal μ-a.e. on X.

(5) Let $(X,\mathscr{M})$ be a measurable space, and let μ,ν be σ-finite measures on $(X,\mathscr{M})$ with $\nu \ll \mu$. Show that there exists a function $f \in L^1(X,\mathscr{M},\mu)$ such that for every $g \in L^1(X,\mathscr{M},\nu)$ and every $E \in \mathscr{M}$ we have

$$\int_E g d\nu = \int_E g f d\mu$$

(6) Justifying all steps, evaluate

$$\int_1^0 \int_y^1 x^{-3/2} \cos\left(\frac{\pi y}{2x}\right) dxdy$$

(7) Let $1 < p < \infty$ and $f \in L^p[0,\infty)$.

(a) Show that for $x > 0$, we have $\left| \int_0^x f(t)dt \right| \leq \|f\|_p x^{1-\frac{1}{p}}$.

(b) Show that

$$\lim_{x\to\infty} \frac{1}{x^{1-\frac{1}{p}}} \int_0^x f(t)dt = 0.$$

Hint: Consider first the case where f has bounded support.

(8) Let $\mathscr{F}$ be the set of all real-valued functions defined on $[0,1]$ which are of the form

$$f(x) = \sum_{n=1}^{\infty} c_n \cos(nx)$$

where the c_n are real numbers satisfying $|c_n| \leq 1/n^3$ for all $n \in \mathbb{N}$. Prove that any sequence of functions in $\mathscr{F}$ has a subsequence that converges uniformly on $[0,1]$.