Real Analysis Qualifying Exam – January 2022

NOTATION: $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real and natural numbers, respectively. Unless otherwise specified, the standard metrics/topologies/measures are always assumed for all spaces involved. m denotes Lebesgue measure on the real line. As customary, the integral of f with respect to Lebesgue measure may be written as $\int f(x)dx$ instead of $\int f dm$. $L^p(X, \mathscr{M}, \mu)$ denotes the space of $\mathscr{M}$-measurable functions f such that $|f|^p$ is μ-integrable, and the norm of f in this space is denoted $\|f\|_p$ (with the necessary modification in the case $p = \infty$).

There are 8 equally-weighted problems in this exam. Answer as many of them as you can.

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function. For $t \in \mathbb{R}$, define $f_t : \mathbb{R} \to \mathbb{R}$ by $f_t(x) = f(t+x)$ for each $x \in \mathbb{R}$. Show that f_t is a measurable function.

(2) Let $f \in L^1(\mathbb{R})$. Suppose that $\int_a^b f(x)dx = 0$ for all rational numbers $a < b$. Prove that f is equal to 0 almost everywhere on $\mathbb{R}$.

(3) Let a and b be real numbers satisfying $a > b > 1$. For each $n \in \mathbb{N}$, let

$$f_n(x) = \frac{n|\cos(x)|}{1 + n^a x^b}.$$

Show that $\lim_{n\to\infty} \int_0^\infty f(x)dx$ exists and find its value.

(4) Give an example of a sequence $(f_n)_{n=1}^\infty$ in $L^1(\mathbb{R})$ such that $\lim_{n\to\infty} \|f_n\|_1 = 0$, but for each $x \in \mathbb{R}$ we have $\limsup_{n\to\infty} f_n(x) = \infty$.

(5) Let $(X, \mathscr{M})$ be a measurable space, and let μ, ν, λ be σ-finite measures on $(X, \mathscr{M})$ with $\nu \ll \mu \ll \lambda$. Show that we have

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda}, \qquad \lambda\text{- almost everywhere on } X.$$

(6) Let $f : (0,1) \to \mathbb{R}$ be Lebesgue integrable. For $x \in (0,1)$ define $g(x) = \int_x^1 \frac{f(t)}{t} dt$. Prove that g is Lebesgue integrable on $(0,1)$, and that

$$\int_0^1 g(x)dx = \int_0^1 f(x)dx$$

Hint: Notice that $g(x)$ can also be written as $g(x) = \int_0^1 \frac{f(t)}{t}\chi_{(x,1)}(t)\,dt$

(7) Let $E \subset \mathbb{R}$ be a measurable set of finite measure. Let $(f_n)_{n=1}^{\infty}$ be a sequence in $L^2(E)$ converging in measure to a function f, and suppose that $\|f_n\|_2 \leq 1$ for each $n \in \mathbb{N}$.

(a) Prove that $f \in L^2(E)$.

(b) Show that $\lim_{n\to\infty} \int_E |f - f_n| dm = 0$.

(8) Let $f : [0,1] \to \mathbb{R}$ be a differentiable function, whose derivative f' is continuous on $[0,1]$. Given $\varepsilon > 0$, prove that there exists a polynomial p such that

$$\|f - p\|_\infty + \left\|f' - p'\right\|_\infty < \varepsilon.$$