Real Analysis
Spring, 2023

Qualifying Exam

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General Instructions: You have three hours and you may use as much paper as you need. Please try to write neatly and concisely, justify every step and state all theorems that you are using. Prominently label the problem number next to each solution.

1

Suppose $\{f_n\}_{n=1}^{\infty}$ and f are non-negative Lebesgue measurable functions on $[0,1]$ and $f_n(x) \to f(x)$ almost everywhere.

(a) Prove that f_n converges to f in measure.

(b) Given an example that shows that one does not necessarily have $\lim\limits_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx$

2

Suppose $f \in L^1([0,1])$. Show that $x^n f \in L^1([0,1])$ for all $n \in \mathbb{N}$, and that

$$\int_0^1 x^n f(x)\,dx \to 0.$$

3

Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Suppose there is a sequence $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$ such that the sequence of functions χ_{A_n} converges in $L^1(X, \mathcal{M}, \mu)$ to a function f. Prove that there exists $A \in \mathcal{M}$ such that f and χ_A are equal μ-a.e. on X.

4

Let $f(x)$ be a non-decreasing function on $[0,1]$ (so that f is differentiable almost everywhere).

(a) Prove that

$$\int_0^1 f'(x)\,dx \le f(1) - f(0)$$

Hint: Fatou.

(b) Give a continuous example of $f(x)$ where the inequality above is strict.

5

For $x \in \mathbb{R}^n$, denote by $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ the usual Euclidean norm. Show that

$$\int_{\mathbb{R}^n} e^{-|x|^2}\, dx = \pi^{n/2}$$

Hint: For $n = 1$, use Fubini's theorem to write

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx \right)^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2 - y^2}\, dx\, dy$$

and use Polar coordinates. For $n > 1$, use the formula $e^{-|x|^2} = e^{-x_1^2} \cdots e^{-x_n^2}$ and Fubini's theorem to reduce to the case $n = 1$.

6

Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) = 1$, and f and g positive measurable functions on X such that $f(x) \cdot g(x) \geq 1$ for μ–a.e. $x \in X$. Show that

$$\int_X f\, d\mu \cdot \int_X g\, d\mu \geq 1.$$

7

(a) State Hölder's inequality.

(b) Prove the following generalization of Hölder's inequality: for $p_1, \ldots, p_n$, $n \geq 2$, $\frac{1}{r} = \sum_{i=1} \frac{1}{p_i}$, with $r, p_i \geq 1$. Then

$$||f_1 \cdots f_n||_r \leq ||f_1||_{p_1} \cdots ||f_n||_{p_n}.$$

8

Give an example of a bounded, continuous function f on $(0, \infty)$ such that $\lim_{x \to \infty} f(x) = 0$ but $f \notin L^p(0, \infty)$ for any $p > 0$.