ANALYSIS QUALIFYING EXAM – FALL 2023

NAME: ……………………………………………………………

Complete 5 of the problems below. If you attempt more than 5 questions, then clearly indicate which 5 should be graded.
(Each problem will count for 10 points.)

For full credit you must provide complete arguments. You are allowed to (and you should) refer to results we discussed in class – do not reprove basic textbook material – but clearly indicate/cite the results you use.

A note on the notation used below: The symbol $\mathscr{L}^p$ refers to the set of all measurable real-valued functions f such that $|f|^p$ is integrable [the value $p = \infty$ is a bit different, of course], and $|f|_{\mathscr{L}^p}$ or $|f|_p$ denote the (pseudo–)norm of $f \in \mathscr{L}^p$. Clearly, the setup implicitly assumes a measure space $(\Omega, \mathscr{F}, \mu)$, so that the domain of f is Ω, measurable means $\mathscr{F}$–measurable, and integrable means integrable with respect to μ. Sometimes we write $\mathscr{L}^p(0,\infty)$ or alike to explicitly indicate parts of the underlying measure space – in this example $\Omega = (0,\infty)$. As is common practice, integration with respect to the Lebesgue measure, denoted by λ, is also written simply as $\int f(x)\,dx$ instead of $\int f(x)\,\lambda(dx)$.

Given a measure μ on some $(\Omega, \mathscr{F})$ and a suitable function $f\colon \Omega \to \mathbb{R}$ we denote the integral of f with respect to μ by any of the following:

$$\mu(f) = \int f(\omega)\,\mu(d\omega) = \int_\Omega f(\omega)\,\mu(d\omega) = \int_\Omega f(\omega)\,d\mu(\omega) = \int_\Omega f\,d\mu = \int f\,d\mu\ .$$

For any two subsets A, B of some set Ω the expression $A \triangle B$ denotes the symmetric difference of the two sets, i.e. $A \triangle B = A \setminus B \cup B \setminus A$.

As is common practice we often suppress the input variables of functions when indicating certain sets of input values. For example, if $f\colon \Omega \to \mathbb{R}$ is some function and $a \in \mathbb{R}$ is some real number, then the set $\{f > a\}$ is shorthand for $\{\omega \in \Omega : f(\omega) > a\}$. To give yet another example, $\mu\{|f| > a\}$ is shorthand for $\mu(A)$ with $A = \{\omega \in \Omega : |f(\omega)| > a\}$.

Given $A \subset \Omega$ we use $\mathbf{1}_A$ to denote the corresponding indicator function, i.e. $\mathbf{1}_A(\omega) = 1$ whenever $\omega \in A$ and $\mathbf{1}_A(\omega) = 0$ whenever $\omega \notin A$.

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1. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, and $f \colon \Omega \to \mathbb{R}$ bounded measurable. Suppose that there exits $C > 0$ and $0 < \alpha < 1$ such that $\mu\{|f| > s\} \leq C\, s^{-\alpha}$ holds for all $s > 0$.
   Prove that $f \in \mathscr{L}^1$.

2. Consider the Lebesgue measure λ on the Borel sets of $[0, 1]$. Let $f \colon [0, 1] \to \mathbb{R}$ be continuous, and let $g \colon [0, 1] \to \mathbb{R}$ be a measurable such that $0 \leq g(x) \leq 1$ for λ–almost all $x \in [0, 1]$.
   Show that the limit $\lim_{n\to\infty} \int_0^1 f(g(x)^n)\, dx$ exists and evaluate it.

3. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space with $\mu(\Omega) = 1$. For any two $A, B \in \mathscr{F}$ define $d(A, B) {:=} \mu(A \triangle B)$. Note that $d(A, B) = \int |\mathbf{1}_A(\omega) - \mathbf{1}_B(\omega)|\, \mu(d\omega) = |\mathbf{1}_A - \mathbf{1}_B|_{\mathscr{L}^1}$, which shows that d defines a (pseudo–)metric on $\mathscr{F}$.
   Prove that in the (pseudo–)metric space $(\mathscr{F}, d)$ for every Cauchy sequence $(A_n)_{n\in\mathbb{N}}$ there exists $A \in \mathscr{F}$ such that $\lim_{n\to\infty} d(A_n, A) = 0$.
   [Hint: You might want to use some basic facts about $\mathscr{L}^1$, but you can also provide a direct proof.]

4. Let $(\Omega, \mathscr{F}, \mu)$ be some measure space with $\mu(\Omega) = 1$, and let f and $(f_n)_{n\in\mathbb{N}}$ be measurable functions from Ω to $\mathbb{R}$. Furthermore, let $\varphi \colon [0, \infty) \to [0, 1]$ be any continuous and strictly increasing function with $\varphi(0) = 0$ and $\lim_{x\to\infty} \varphi(x) = 1$.
   Prove that if f_n converges to f in μ–measure then $\displaystyle\lim_{n\to\infty} \int_\Omega \varphi(|f_n(\omega) - f(\omega)|)\, \mu(d\omega) = 0$.

5. Let $E \subset \mathbb{R}^n$ be a Borel set with $\lambda(E) > 0$, where λ denotes the Lebesgue measure.
   Show that for every $\epsilon > 0$ there exists a ball $B \subset \mathbb{R}^n$ such that $\lambda(E \cap B) \geq (1 - \epsilon)\, \lambda(B)$.

6. Suppose that $(\Omega, \mathscr{F}, \mu)$ measure space with $\mu(\Omega) = 1$. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of measurable sets such that $\mu(A_n) \geq \frac{1}{2}$ for all $n \in \mathbb{N}$.
   Prove or disprove by counterexample: The set $A = \cap_k \cup_{n\geq k} A_n$ of all those $\omega \in \Omega$ which belong to A_n for infinitely many $n \in \mathbb{N}$ has measure $\mu(A) \geq \frac{1}{2}$.

7. Prove that for every $\epsilon > 0$ there exists an open set $O_\epsilon \subset \mathbb{R}$ such that $\mathbb{Q} \subset O_\epsilon$ and $\lambda(O_\epsilon) \leq \epsilon$, where λ denotes the Lebesgue measure on $\mathbb{R}$.

8. Let $(\Omega, \mathscr{F}, \mu)$ be any measure space, and let ν be any measure defined on the Borel sets $\mathcal{B}$ of $[0, \infty)$. [In particular, μ and ν are not assumed to be necessarily σ–finite.]
   Prove that
   $$\int_\Omega \nu[0, f(\omega))\, \mu(d\omega) = \int_{[0,\infty)} \mu\{f > t\}\, \nu(dt)$$
   holds true for any measurable function $f \colon \Omega \to [0, \infty)$. Be sure to address any issues relating to μ and ν not necessarily being σ–finite, should there be any at all.

9. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space with $\mu(\Omega) = 1$, and suppose that there is an algebra $\mathscr{F}_0 \subset \mathscr{F}$ such that $\sigma(\mathscr{F}_0) = \mathscr{F}$.
   Prove that for every $A \in \mathscr{F}$ and every $\epsilon > 0$ there exists $A_\epsilon \in \mathscr{F}_0$ such that $\mu(A \triangle A_\epsilon) \leq \epsilon$.

10. Consider $\Omega = [0, 1]$ with its Borel σ–algebra $\mathscr{F}$. Let λ be the Lebesgue measure on $(\Omega, \mathscr{F})$, and let μ be the counting measure on $(\Omega, \mathscr{F})$ [i.e. $\mu(A)$ is the number of elements of A if $A \in \mathscr{F}$ is a finite set, otherwise $\mu(A) = \infty$].
    Prove that $\lambda \ll \mu$, but there exists no measurable $f \geq 0$ such that $\lambda(A) = \int \mathbf{1}_A(\omega)\, f(\omega)\, \mu(d\omega)$ for all $A \in \mathscr{F}$. Explain very briefly why this is not contradicting the Radon-Nikodym theorem.