Qualifying exam Topology August 2023

Name: ....

(a) Problems 6(a), 9: just the answers are sufficient, no explanations are required;
(b) All other problems: please provide proofs for all your answers and claims;
(c) All subsets of $\mathbb{R}^n$ are given the subspace topology unless stated otherwise.

1. **(2+2 points)** Consider the real line with the standard topology (this space will be denoted by $\mathbb{R}$) and with the topology with basis $\mathcal{B} = \{[a, b) : a < b\}$ (denoted by $\mathbb{R}_\ell$). Are the following functions continuous?
   (a) $f : \mathbb{R} \to \mathbb{R}_\ell, \quad f(x) = x;$
   (b) $g : \mathbb{R}_\ell \to \mathbb{R}, \quad g(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}.$

2. **(2+2 points)** (a) Is $\mathbb{Q}$ a quotient of $\mathbb{Z}$ (that is, is there an equivalence relation $\sim$ on $\mathbb{Z}$ such that $\mathbb{Q} \cong \mathbb{Z}/\sim$, or, equivalently, is there a quotient map $p : \mathbb{Z} \to \mathbb{Q}$)?

   (b) Is $\mathbb{Z}$ a quotient of $\mathbb{Q}$?

3. **(3+2 points)** Let X be a Hausdorff space that is locally compact but not compact, and let Y be its one-point compactification. (a) Show that if X is connected, then so is Y.
   (b) Is the converse also true? Give a proof or a counterexample.

4. **(2+2+2 points)** Let $X = (-1, 1) \subseteq \mathbb{R}$. (a) Find a metric on X that generates the standard topology and makes X a complete space.
   (b) Find a metric on X that generates the standard topology and makes X an incomplete space.
   (c) Is there such a metric, as in part (b), on $Y = [-1, 1]$?

5. **(2+2+2+2 points)** Find the fundamental groups $\pi_1(X, x)$ of the following spaces. *Suggested method:* Find a familiar space that is a deformation retract of X.
   (a) a cylinder X, defined as the quotient of the square $Q = \{(x, y) \in \mathbb{R}^2 : -1 \leq x, y \leq 1\}$ by the equivalence relation $(-1, y) \sim (1, y)$;

(b) a Möbius band X, defined as the quotient of Q by $(-1, y) \sim (1, -y)$;
(c) $X = \mathbb{R}^3 \setminus \{A_x \cup A_y \cup A_z\}$, where $A_x = \{(a, 0, 0) : a \geq 0\}$ denotes the non-negative x-axis etc.;
(d) the punctured projective plane $X = P^2 \setminus \{p\}$

6. **(1+4 points)** Write $p = (0, 0)$, $x = (2, 0)$, $q = (3, 0)$, and let $X = \mathbb{R}^2 \setminus \{p\}$. (a) What is $\pi_1(X, x)$?

   (b) Explain how the result form part (a) can also be obtained by applying the Seifert-van Kampen theorem to the decomposition $X = U \cup V$, $U = X \setminus \{q\}$, $V = B(q, 2)$ (the open disk of radius 2 about q).

7. **(4 points)** Show that every continuous map $f : P^2 \to S^1$ is nulhomotopic (= homotopic to a constant map). *Suggestion:* Try to lift f to the universal cover of S^1.

8. **(4 points)** Denote the closed unit disk by $\overline{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, and let $p : \overline{D} \to X$ be a covering map. Show that X is homeomorphic to $\overline{D}$. *Suggestion:* Investigate the group of covering transformations.

9. **(4 points)** True or false? Four correct answers are sufficient for full credit here, so you don't have to answer every question. However, I will deduct one point for each incorrect answer in excess of one, so please be careful taking guesses.

	True	False
If X can be embedded in Y and Y can be embedded in X, then X and Y are homeomorphic.		
Denote the projection maps from a product space by $p_\alpha : \prod_\beta X_\beta \to X_\alpha$. If $p_\alpha(U)$ is open in X_α for all α, then U is open in the product topology.		
If X is a compact Hausdorff space, then every quotient of X satisfies T_1 (= points are closed).		
If X has infinitely many connected components, then there is a continuous surjective map $f : X \to \mathbb{Z}$.		
If X is connected, then X is locally connected.		
Let $p : E \to B$ be a covering map. If $p(x) = p(y)$, then there is a unique covering transformation $\varphi : E \to E$ with $\varphi(x) = y$.		
Let X be path connected and locally path connected. If X has a universal cover, then given any $x \in X$ and any neighborhood V of x, there is a simply connected neighborhood $U \subseteq V$ of x.		