ALGEBRA QUALIFYING EXAM - JANUARY 2024

Instructions

- Do as many problems as the time allows.
- Each problem is worth the same, but a complete solution of one problem is worth more than partial work on two problems.
- All rings are assumed to have a 1.
- Good luck!

Problems

1. Let G be a group and let $G' = [G, G]$ be the commutator subgroup of G, i.e., the subgroup generated by $\{[a, b] := aba^{-1}b^{-1} | a, b \in G\}$.
   (a) Prove that G' is normal in G.
   (b) Show that $A = G/G'$ is abelian.
   (c) Let H be an abelian group and consider a group homomorphism $\varphi : G \to H$. Show that φ factors through A. That is, show that there exists a unique homomorphism $f : A \to H$ such that $\varphi = f \circ \pi$ where $\pi : G \to G/G' = A$ is the natural projection homomorphism.

2. Let G be a group of order 55.
   (a) Show that G has a normal subgroup of order 11.
   (b) For G non-abelian, how many elements of order 5 does G have?
   (c) Classify the groups of order 55.

3. Let K/F be a finite extension of fields. Assume further that K/F is a Galois extension and set $G = \mathrm{Gal}(K/F)$. For $\alpha \in K$, let $G_\alpha = \{\sigma \in G : \sigma(\alpha) = \alpha\}$.
   (a) Recall from Galois theory that $K/F(\alpha)$ is a Galois extension (for all $\alpha \in K$). Show that $\mathrm{Gal}(K/F(\alpha)) = G_\alpha$.
   (b) For $\alpha, \beta \in K$, explain why $F(\alpha) = F(\beta)$ if and only if $G_\alpha = G_\beta$.
   (c) For $\alpha \in K$, prove that $F(\alpha)/F$ is a Galois extension if and only if each $\sigma \in G_\alpha$ satisfies
   $$\sigma(\tau(\alpha)) = \tau(\alpha), \quad \forall\, \tau \in G.$$

4. (a) For F a field, show that the polynomial ring $F[\mathrm{X}, \mathrm{Y}]$ is not principal (that is, not every ideal admits a generator).
   (b) Suppose a commutative ring R contains a unique maximal ideal $\mathfrak{m}$. Such rings are called *local*. Write $R^\times$ for the group of units of R, so that
   $$R^\times = \{r \in R : \exists\, s \in R \text{ such that } rs = 1\}.$$
   Show that $1 + \mathfrak{m} = \{1 + x : x \in \mathfrak{m}\}$ is a subgroup of $R^\times$. In particular, you need to show that $1 + \mathfrak{m} \subseteq R^\times$.

5. Let R be a commutative ring and let I be an ideal in R.
   (a) Show that if R is Noetherian then the quotient ring R/I is Noetherian.
   (b) Show that if the polynomial ring $R[\mathrm{X}]$ is Noetherian then R is Noetherian.
   (c) Give an example with proof of a commutative ring that is not Noetherian.

6. Consider the following rings:

$$R_1 = \mathbb{Z}[\sqrt{2}],\ R_2 = \mathbb{Z}[\sqrt{3}],\ R_3 = \mathbb{Z}[1/2],$$

   viewed say as subrings of $\mathbb{C}$.
   (a) Show that R_1 and R_2 are not isomorphic as rings.
   (b) Show that R_1 and R_2 are isomorphic as $\mathbb{Z}$-modules (equivalently, as abelian groups).
   (c) Show that R_1 and R_3 are not isomorphic as $\mathbb{Z}$-modules.