ANALYSIS QUALIFYING EXAM – JANUARY 2024

NAME: ……………………………………………………………

Complete 5 of the problems below. If you attempt more than 5 questions, then clearly indicate which 5 should be graded.
(Each problem will count for 10 points.)

For full credit you must provide complete arguments. You are allowed to (and you should) refer to results we discussed in class – do not reprove basic textbook material – but clearly indicate/cite the results you use and make sure you verify any conditions that need to be satisfied.

A note on the notation used below: The symbol $\mathscr{L}^p$ refers to the set of all measurable real-valued functions f such that $|f|^p$ is integrable [the value $p = \infty$ is a bit different, of course], and $|f|_{\mathscr{L}^p}$ or $|f|_p$ denote the (pseudo–)norm of $f \in \mathscr{L}^p$. Clearly, the setup implicitly assumes a measure space $(\Omega, \mathscr{F}, \mu)$, so that the domain of f is Ω, measurable means $\mathscr{F}$–measurable, and integrable means integrable with respect to μ. Sometimes we write $\mathscr{L}^p(0,\infty)$ or alike to explicitly indicate parts of the underlying measure space – in this example $\Omega = (0,\infty)$. As is common practice, integration with respect to the Lebesgue measure, denoted by λ, is also written simply as $\int f(x)\,dx$ instead of $\int f(x)\,\lambda(dx)$.

Given a measure μ on some $(\Omega, \mathscr{F})$ and a suitable function $f\colon \Omega \to \mathbb{R}$ we denote the integral of f with respect to μ by any of the following:

$$\mu(f) = \int f(\omega)\,\mu(d\omega) = \int_\Omega f(\omega)\,\mu(d\omega) = \int_\Omega f(\omega)\,d\mu(\omega) = \int_\Omega f\,d\mu = \int f\,d\mu\ .$$

For any two subsets A, B of some set Ω the expression $A \triangle B$ denotes the symmetric difference of the two sets, i.e. $A \triangle B = A \setminus B \cup B \setminus A$.

As is common practice we often suppress the input variables of functions when indicating certain sets of input values. For example, if $f\colon \Omega \to \mathbb{R}$ is some function and $a \in \mathbb{R}$ is some real number, then the set $\{f > a\}$ is shorthand for $\{\omega \in \Omega : f(\omega) > a\}$. To give yet another example, $\mu\{|f| > a\}$ is shorthand for $\mu(A)$ with $A = \{\omega \in \Omega : |f(\omega)| > a\}$.

Given $A \subset \Omega$ we use $\mathbf{1}_A$ to denote the corresponding indicator function, i.e. $\mathbf{1}_A(\omega) = 1$ whenever $\omega \in A$ and $\mathbf{1}_A(\omega) = 0$ whenever $\omega \notin A$.

Given a collection C of subsets of some set Ω we use the notation $\sigma(C)$ to denote the smallest σ–algebra that contains all sets in C.

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1. Let $\varphi\colon [a,b] \to \mathbb{R}$ be an absolutely continuous and increasing (not necessarily strictly so) function. Set $A{:=}\varphi(a)$ and $B{:=}\varphi(b)$.
   Prove that for any Lebesgue integrable $f\colon [A,B] \to \mathbb{R}$ the substitution rule
   $$\int_{[A,B]} f(x)\,\lambda(dx) = \int_{[a,b]} f(\varphi(t))\,\varphi'(t)\,\lambda(dt)$$
   holds.

2. Let M be a metric space. Suppose that $\mathcal{A}$ is an algebra of subsets of M such that every $E \in \mathcal{A}$ is simultaneously open and compact.
   Prove that every finitely additive $\mu\colon \mathcal{A} \to [0,\infty]$ with $\mu(M) = 1$ extends uniquely to a measure on $\sigma(\mathcal{A})$.

3. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space with $\mu(\Omega) = 1$.
   Prove that for every $f \in \mathscr{L}^\infty$ the limit $\lim_{p\to\infty} |f|_p$ exists and is equal to $|f|_\infty$.

4. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, and let f and $(f_n)_{n\in\mathbb{N}}$ be integrable functions such that $\lim_{n\to\infty} f_n = f$ holds μ–almost everywhere.
   Prove that $(f_n)_{n\in\mathbb{N}}$ converges to f in $\mathscr{L}^1$ if and only if $\int_\Omega |f_n(\omega)|\,\mu(d\omega)$ converges to $\int_\Omega |f(\omega)|\,\mu(d\omega)$.

5. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, and suppose that $f\colon (0,1)\times\Omega \to \mathbb{R}$ satisfies
   (a) For every $x \in (0,1)$ fixed the function $f(x,.)$ is integrable.
   (b) The partial derivative $\frac{\partial f}{\partial x}(x,\omega)$ exists for all (x,ω).
   (c) There exist $\epsilon > 0$ and $g\colon \Omega \to \mathbb{R}$ integrable such that for all $t_1, t_2 \in (0,1)$ with $|t_2 - t_1| < \epsilon$ the inequality $|\frac{f(t_2,\omega)-f(t_1,\omega)}{t_2-t_1}| \le g(\omega)$ holds for μ–almost all $\omega \in \Omega$.
   Prove that $F\colon (0,1) \to \mathbb{R}$ defined by $F(x){:=}\int_\Omega f(x,\omega)\,\mu(d\omega)$ is differentiable at every x, and that $F'(x) = \int_\Omega \frac{\partial f}{\partial x}(x,\omega)\,\mu(d\omega)$ holds.

6. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space with $\mu(\Omega) = 1$, and let $\mathscr{G} \subset \mathscr{F}$ be a sub–σ-algebra. The restriction of μ to $\mathscr{G}$ is a measure on $(\Omega, \mathscr{G})$. To make clear the measure space involved let $\mathscr{L}^p_{\mathscr{F}}$ and $\mathscr{L}^p_{\mathscr{G}}$ denote the $\mathscr{L}^p$–spaces corresponding to $(\Omega, \mathscr{F}, \mu)$ and $(\Omega, \mathscr{G}, \mu)$, respectively. It is known (as we showed in class) that for every $f \in \mathscr{L}^1_{\mathscr{F}}$ there exists $f_* \in \mathscr{L}^1_{\mathscr{G}}$ such that
   - $|f_*|_1 \le |f|_1$ i.e. $\int_\Omega |f_*(\omega)|\,\mu(d\omega) \le \int_\Omega |f(\omega)|\,\mu(d\omega)$
   - $\int_\Omega f(\omega)\,g(\omega)\,\mu(d\omega) = \int_\Omega f_*(\omega)\,g(\omega)\,\mu(d\omega)$ for all $g \in \mathscr{L}^\infty_{\mathscr{G}}$,

   Prove that if $f \in \mathscr{L}^2_{\mathscr{F}}$, then
   (a) $f_* \in \mathscr{L}^2_{\mathscr{G}}$, and $|f_*|_2 \le |f|_2$.
   (b) $f - f_* \perp \mathscr{L}^2_{\mathscr{G}}$, i.e. the inner product (taken in $\mathscr{L}^2_{\mathscr{F}}$) of $f - f_*$ and every $h \in \mathscr{L}^2_{\mathscr{G}}$ vanishes.

7. Let $(\Omega, \mathscr{F})$ be a measurable space such that there exists a countable (possibly finite) collection $(A_n)_{n\in I}$, $I \subset \mathbb{N}$, of subsets of $\mathscr{F}$ such that $\mathscr{F} = \sigma(A_n : n \in I)$.
   Prove that for every $\omega \in \Omega$ the set $[\omega]{:=}\bigcap_{A\in\mathscr{F}\,:\,\omega\in A} A$ is an element of $\mathscr{F}$ by establishing that $[\omega] = \Big(\bigcap_{n\in I\,:\,\omega\in A_n} A_n\Big) \cap \Big(\bigcap_{n\in I\,:\,\omega\in A_n^c} A_n^c\Big)$.

8. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space with $\mu(\Omega) = 1$. For any measurable function $f\colon \Omega \to [-\infty, \infty]$ introduce
$$F\colon (-\infty, \infty) \to [-\infty, \infty]\,, \qquad F(\beta) := \log\Big(\int_\Omega e^{\beta f(\omega)}\,\mu(d\omega)\Big)$$
where the logarithm and the exponential functions are extended by continuity to all of $[0, \infty]$ and $[-\infty, \infty]$, respectively [i.e. we use $\log(0) = -\infty$, $\log(\infty) = \infty$, $e^{-\infty} = 0$, $e^{\infty} = \infty$].
   (a) Explain VERY briefly why F is well-defined (assuming only measurability of f), and verify that $F(0) = 0$.
   (b) Prove that F is lower semi-continuous, i.e. prove that $F(\beta_0) \le \liminf_{\beta\to\beta_0} F(\beta)$ holds for every $\beta_0 \in \mathbb{R}$.
   (c) Prove that F is a convex (aka concave-up) function.

9. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and $f\colon \Omega \to [0, \infty]$ measurable. Define $G_f\colon [0, \infty) \to [0, \infty]$ by $G_f(s) := \mu\{f > s\}$ for all $s \in [0, \infty)$.
Prove that
$$\int_\Omega f(\omega)^2\,\mu(d\omega) = \int_0^\infty \int_0^\infty \min\{G_f(s), G_f(t)\}\,ds\,dt\,.$$
Note: Be sure to address any issues relating to μ not necessarily being σ–finite, should there be any at all. If you can only prove this result assuming σ–finiteness of μ, then you may, but you must point out where and why exactly you need to make this additional assumption.

10. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space with $\mu(\Omega) = 1$, and suppose that there is an algebra $\mathcal{A} \subset \mathscr{F}$ such that $\sigma(\mathcal{A}) = \mathscr{F}$.
Prove that for every $A \in \mathscr{F}$ and every $\epsilon > 0$ there exists $A_\epsilon \in \mathcal{A}$ such that $\mu(A \triangle A_\epsilon) \le \epsilon$.