ALGEBRA QUALIFYING EXAM - AUGUST 2024

Instructions

- **Complete 6 problems including at least 1 problem from each of the 4 sections.**
- You are welcome to complete more than 6 problems if you can't choose which 6 to do or want to demonstrate some additional understanding.
- All rings are assumed to have a 1.
- Good luck!

Problems

I. Groups.

1. (a) Let G be a group of finite order n, and let m be an integer coprime to n. Suppose $g, h \in G$ with $g^m = h^m$. Show that $g = h$.
   (b) Let G be a group, and let H be a normal subgroup of index p. Let K be any subgroup of G which is not contained in H. Describe $K/(K \cap H)$. Justify your answer.

2. Let p be a prime. Let S be a set of cardinality p^b for some positive integer b. Suppose G is a finite group that acts transitively on S (i.e., if $s, t \in G$, then there exists some $g \in G$ such that $gs = t$). Let P be a Sylow p-subgroup of G. Prove that P acts transitively on S.

II. Rings.

3. Let R be a commutative ring, and let X be the set of prime ideals in R. Given a subset $A \subset R$, we define $V(A) \subset X$ to be the set of prime ideals in R which contain A. That is,

$$V(A) = \{P \in X : P \supset A\}.$$

   (a) Describe $V(\{0\})$ and $V(\{1\})$.
   (b) Given subsets $A, B \subset R$, denote

$$AB := \{ab \in R | a \in A, b \in B\}.$$

   Prove that $V(AB) = V(A) \cup V(B)$.

4. (a) Is $\mathbb{Q}[W, X, Y, Z]$ a UFD (unique factorization domain)? Briefly justify your answer.
   (b) Find an element of $R = \mathbb{Q}[W, X, Y, Z]/(WX - YZ)$ which is irreducible but not prime. Briefly justify your answer.
   (c) Is $R = \mathbb{Q}[W, X, Y, Z]/(WX - YZ)$ a UFD? Briefly justify your answer.
   (d) Show that the ideal $I = (X, Y)$ in $R = \mathbb{Q}[W, X, Y, Z]/(WX - YZ)$ is prime.

III. Modules.

5. (a) Let R be a commutative Noetherian ring. Prove that every surjective R-module homomorphism $\phi : R \to R$ is an automorphism.
   (b) If R is any integral domain with quotient field F, show that $(F/R) \otimes_R (F/R) = 0$.

6. Let A be a matrix over a field F with characteristic polynomial $ch_A(x) = x^3 - 1$.
   (a) For $F = \mathbb{C}$, give the rational canonical and Jordan canonical forms for A.
   (b) What are the possible Jordan canonical forms for A when $F = \mathbb{Z}/3\mathbb{Z}$?

IV. Fields.

7. Let $f(x) = x^3 + 4x + 2 \in F[x]$ for a field F. Let K/F denote the splitting field of $f(x)$.
   (a) Show that $f(x)$ is irreducible for:
      (i) $F = \mathbb{Q}$,
      (ii) $F = \mathbb{F}_5$.
   (b) Describe $\mathrm{Gal}(K/F)$ for $F = \mathbb{Q}$. (Hint: It might help to find how many real roots there are. You can do this using calculus).
   (c) For $F = \mathbb{F}_5$, describe K and $\mathrm{Gal}(K/F)$. Here, you should describe $\mathrm{Gal}(K/F)$ both as an abstract group and in terms of its action on K.

8. Fix a prime $p \geq 5$. Let $f(x) \in \mathbb{Q}[x]$ be irreducible with degree p. Let $K/\mathbb{Q}$ be a splitting field for $f(x)$.
   (a) Show that p divides $[K : \mathbb{Q}]$.
   (b) How do we know the following two facts:
      (i) $f(x)$ is separable.
      (ii) $K/\mathbb{Q}$ is Galois.
   (c) From now on, assume the polynomial $f(x)$ has exactly $p - 2$ real roots. Show that G contains a transposition (when viewed as a subgroup of S_p via the action on roots of $f(x)$).
   (d) Briefly show that G contains a p-cycle (when viewed as a subgroup of S_p).
   (e) Is $f(x)$ solvable by radicals? Briefly justify your answer.