University of Oklahoma Department of Mathematics
Real Analysis Qualifier Exam
August, 2024

Directions: Answer each question on a separate page (or pages), writing your name in the upper right corner of each page and the problem number in the upper left corner of each page. Each problem is worth 10 points. Completely justify your work and state which theorems or results you are citing and how you have confirmed the hypotheses of those theorems. You have 3 hours to complete this exam.

Notation: Lebesgue measure is denoted by λ, the real line is denoted by $\mathbb{R}$, integration with respect to Lebesgue measure may be denoted with $\mathrm{d}\lambda$ or just $\mathrm{d}x$.

We will use $\mathcal{S}$ to indicate an arbitrary σ-algebra on a set X. In the case when $X \subseteq \mathbb{R}$, we will use $\mathcal{B}$ to indicate the Borel σ-algebra and $\mathcal{L}$ to indicate the Lebesgue σ-algebra.

The space $\mathcal{L}^p(\mu)$ indicates the functions with finite p-norm on an abstract measure space $(X, \mathcal{S}, \mu)$. When we write $\mathcal{L}^p(E)$, where $E \subseteq \mathbb{R}$, we mean the $\mathcal{L}^p$ functions on E with respect to Lebesgue measure.

It may be useful to recall the limit $\displaystyle\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ for all $x \in \mathbb{R}$.

1. Suppose $(X, \mathcal{S}, \mu)$ is a measure space and (f_k) is a sequence of real-valued μ-measurable functions on X. Suppose that
$$\sum_{k=1}^{\infty} \int_X |f_k| \, \mathrm{d}\mu < \infty.$$
Prove that for μ-almost every $x \in X$, $\sum_{k=1}^{\infty} f_k(x)$ converges to a (finite) real number.

2. Let $0 < p < p' < \infty$
   a) (*5 points*) Prove that $\mathcal{L}^p([0,1]) \supseteq \mathcal{L}^{p'}([0,1])$.
   b) (*5 points*) Prove that neither $\mathcal{L}^p(\mathbb{R})$ nor $\mathcal{L}^{p'}(\mathbb{R})$ is a subset of the other.

3. Compute
$$\lim_{n\to\infty} \int_0^1 \frac{1}{(1+\frac{x}{n})^n x^{\frac{1}{n}}} \, \mathrm{d}x.$$
Explain your reasoning at each step.

4. Let X be a set and (E_k) be a sequence of subsets of X. Define
$$\limsup E_k = \bigcap_k \bigcup_{n\geq k} E_n \quad \text{and} \quad \liminf E_k = \bigcup_k \bigcap_{n\geq k} E_n.$$
   a) (*2 points*) Let $\mathcal{S}$ be a σ-algebra of subsets of X and let $(E_k) \subseteq \mathcal{S}$. Prove that $\limsup E_k$ and $\liminf E_k$ are also elements of $\mathcal{S}$.
   b) (*8 points*) Let $(X, \mathcal{S}, \mu)$ be a finite measure space and let $(E_k) \subseteq \mathcal{S}$. Prove that
$$\mu(\liminf E_k) \leq \liminf \mu(E_k) \leq \limsup \mu(E_k) \leq \mu(\limsup E_k).$$

5. Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are finite measure spaces. Suppose $\lambda : \mathcal{S} \to [0,\infty)$ and $\eta : \mathcal{T} \to [0, \infty)$ are finite measures such that $\lambda \ll \mu$ and $\eta \ll \nu$.
   a. (*8 points*) Prove that $\lambda \times \eta \ll \mu \times \nu$ on the product space $(X \times Y, \mathcal{S} \times \mathcal{T})$.
   b. (*2 points*) Find a formula for the Radon Nikodym derivative $\frac{\mathrm{d}(\lambda\times\eta)}{\mathrm{d}(\mu\times\nu)}$ and justify your answer.

6. Let $((a,b], \mathcal{B}, \mu)$ be a finite measure space and let g_μ be the cumulative distribution function for μ, i.e. $g_\mu(x) = \mu((a,x])$ for all $a < x \leq b$. Prove that μ is absolutely continuous with respect to Lebesgue measure λ if and only if the function g_μ is absolutely continuous on $(a,b]$.

7. **The Counterexample Olympics** On each part below, find a counterexample that proves the statement. Give a brief justification or draw a picture to illustrate your example. Each is worth 2 points.
   (a) Measurable functions (f_k) may converge pointwise almost everywhere on a finite measure space $(X, \mathcal{S}, \mu)$ while
$$\lim_{k\to\infty} \int_X f_k \, \mathrm{d}\mu \neq \int_X f \, \mathrm{d}\mu.$$
   (b) Not every Banach space is a Hilbert space.
   (c) The sigma algebra $\mathcal{L}$ of Lebesgue measurable subsets of $\mathbb{R}$ is not equal to the Borel sigma algebra $\mathcal{B}$ on $\mathbb{R}$.
   (d) Not every (improper) Riemann integrable function is Lebesgue integrable.
   (e) A sequence of measurable functions (f_k) may converge in measure to a measurable function f on the Borel measure space $([0,1], \mathcal{B}, \lambda)$ even though there is *no* $x \in [0,1]$ with $f_k(x)$ converging to $f(x)$. In other words, convergence in measure does not imply pointwise a.e. convergence.