Aug 5, 2024 **TOPOLOGY QUALIFYING EXAM** 9:00-12:00pm

Instructions: You have three hours to complete the exam. There are three parts to the exam; follow the instructions for each part and clearly label the problems you are solving.

I

Definitions and examples. Complete all the problems below. When providing examples, you don't need to give a detailed proof, but make sure to give a clear description of each example.

1. Define what it means for a topological space to be *compact*, *limit-point compact*, and *sequentially compact*. Give one theorem that ties all three properties together.
2. Define what it means for a topological space to be *connected* and *locally-connected*. Give an example of a space that's connected but not locally-connected.
3. Let $q : X \to Y$ be a continuous surjection between two topological spaces. Define what it means for q to be a *quotient map*. Give an example of a continuous surjection which is not a quotient map.
4. Define what it means for a subspace $A \subset X$ to be a *retract* and a *deformation retract* of X. Give an example of a subspace A which is a retract but NOT a deformation retract of X.
5. (a) State the path lifting property for a covering map $p : Y \to X$. (b) Let $p : \mathbb{R}_+ \to S^1$ be defined by $p(t) = (\cos 2nt, \sin 2nt)$; that is p is the restriction of the covering map $\mathbb{R} \to S^1$ to the positive reals $\mathbb{R}_+$. Show that p does not satisfy the lifting property. (This shows that p is not a covering map despite that it's continuous, surjective, and a local-homeomorphism.)

II

Solve THREE of these problems. All proofs should be clear and concise. State clearly any theorems you are using.

1. Consider the set $\mathbb{Z}$ of integer numbers endowed with the cofinite topology (only finite sets and the entire set are closed).
   (a) Is this space compact? Prove your statement.
   (b) Is it connected? Prove your statement.
   (c) Is it T1 (points are closed)? Hausdorff? Regular? Normal? metrizable? Explain.
2. Let $f : \mathbb{R} \to \mathbb{R}^\omega$ be defined by $f(t) = (t, t, t, \ldots)$. Is f continuous when $\mathbb{R}^\omega$ is given the: (a) product topology? (b) box topology?
3. (a) Show that a continuous bijection $f : X \to Y$ from a compact space X to a Hausdorff space Y is a homeomorphism.

(b) Suppose $\mathcal{T}$ is a topology on X which makes X both compact and Hausdorff. Show that one cannot make $\mathcal{T}$ coarser or finer without destroying either that X is compact or that X is Hausdorff.

4. Let X be a topological space and let X^* be the set of connected components of X. Let $p : X \to X^*$ be the natural projection map: $p(x)$ is the connected component of X containing x. Equip X^* with the quotient topology. Show that the connected components of X are open if and only if X^* has the discrete topology.

5. A topological space X has the *fixed point property* if every continuous map $f : X \to X$ has a fixed point (i.e. there exists $x \in X$ such that $f(x) = x$).

   (a) Show that the sphere $S^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$ does not have the fixed point property.

   (b) Suppose that X has the fixed point property. Show that X is T0, that is, for every pair of distinct points in X, at least one of the points has an open neighborhood which does not contain the other point (Hint: suppose otherwise and define a continuous map $f : X \to X$ without fixed points).

III

Solve THREE of these problems. All proofs should be clear and concise. State clearly any theorems you are using.

1. Suppose X is a connected CW complex that has just one cell in dimensions 0 and 2, but may have any number of cells in other dimensions. Also suppose the attaching map of the 2-cell $\varphi : S^1 \to X^1$ is NOT surjective, where X^1 denotes the 1-skeleton of X. Consider these questions about X:

   (a) Is X simply connected?

   (b) Is $\pi_1(X)$ infinite?

   (c) Is $\pi_1(X)$ abelian?

   For each one, if the question can be answered "yes" or "no" based solely on the information provided, give such an answer and justify it. Otherwise, give examples to show that the answer is not determined by the information provided.

2. Prove that the 2-sphere S^2 is not homeomorphic to the 3-sphere S^3. You will need to give a detailed explanation of any statement you are using.

3. Determine the fundamental groups of the following spaces. State any theorems you're using..

   (a) $X = \mathbb{RP}^2 \vee \mathbb{RP}^2$

   (b) $X = S^2 \cup \{(0, 0, z) : -1 \leq z \leq 1\}$, i.e. the 2-sphere together with a diameter.

   (c) X is obtained from removing 1 point from the torus $S^1 \times S^1$.

4. (a) Define what it means to be a covering space and what a lifting of a map is with respect to a covering map.

   (b) Show that given $p : \tilde{X} \to X$ a covering map, every map $f : S^n \to X$ lifts to $\tilde{X}$ provided $n > 1$

   (c) Give an example where the above statement fails when $n = 1$.

5. In the following, $p : Y \to X$ is a covering map, where X is connected and locally path-connected, and Y is connected.

   (a) Give the definition of a deck transformation (also called covering space automorphism) of $p : Y \to X$, and sketch an argument that deck transformations form a group under composition. This group is denoted by $G(Y)$.

   (b) Suppose $p : Y \to X$ is a normal cover. What is $G(Y)$?

   (c) Let $X = S^1 \vee S^1$. Construct a covering space Y with $G(Y) = \mathbb{Z}$.