ALGEBRA QUALIFYING EXAM — JANUARY 2025

Instructions

- Complete 6 of the 7 problems below. If you complete more than 6 problems, your score will be based on your best 6.
- All rings are assumed to have a 1.
- Good luck!

Problems

1. Let G be a group of order 30, and let H and K be subgroups of G with $|H| = 3$ and $|K| = 5$.
   (a) Prove that at least one of H and K is normal.
   (b) Show that G contains a normal subgroup N isomorphic to Z_{15}.
   (c) Deduce that in fact H and K are *both* normal in G.

2. Consider a group G of order 128 acting on a set X with 5 elements.
   (a) Show that there are at least two elements $g_1, g_2 \in G$ with the property that $g_1.x = g_2.x$ for all $x \in X$.
   (b) Prove that this action has a fixed point. I.e., show that there exists an element $x \in X$ such that $g.x = x$ for all $g \in G$.

3. Let $R = \mathbb{Z}[\sqrt{10}]$. Note that
$$6 = 2 \cdot 3 = (4 + \sqrt{10})(4 - \sqrt{10}).$$
   (a) Are 2 and 3 irreducible in R?
   (b) Are 2 and 3 prime in R?
   (c) Is R a UFD?
   Justify your answers.

4. Let R be a ring. Recall that a nontrivial (left) R-module M is called **simple** if the only submodules are 0 and M.
   (a) Let M and N be simple R-modules. Prove that every R-module homomorphism $\varphi \in \mathrm{Hom}_R(M, N)$ is either the zero map or an isomorphism.
   (b) Suppose R is commutative. Show that that every simple R-module M is isomorphic to $R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ in R.

5. Let V be a finite-dimensional vector space over a field F and suppose T is a nonsingular linear transformation of V such that $T^{-1} = T^2 + T$.
   (a) If $F = \mathbb{Q}$, show that the dimension of V is divisible by 3.
   (b) If the dimension of V is precisely 3, prove that all such transformations are similar (over $\mathbb{Q}$), and give an example of a 3×3 matrix A for one such transformation (over $\mathbb{Q}$).
   (c) The matrix A from part (b) has eigenvalues in $\mathbb{C}$. Prove that these eigenvalues are distinct (without computing them).

6. Let $\alpha = \sqrt{2 + \sqrt{3}}$, and let $K = \mathbb{Q}(\alpha)$.
   (a) Show that $[K : \mathbb{Q}] = 4$.
   (b) Find the minimal polynomial $f(x)$ of α over $\mathbb{Q}$. Make sure you justify that this $f(x)$ really is the minimal polynomial.
   (c) Show that K is the splitting field for $f(x)$.
   (d) Describe the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$, up to isomorphism.
   Hint: Different approaches could work in part (d), but it might be helpful to note that $6 = (\alpha + \frac{1}{\alpha})^2 \in K$ and $2 = (\alpha - \frac{1}{\alpha})^2 \in K$, so $\mathbb{Q}(\sqrt{6})$ and $\mathbb{Q}(\sqrt{2})$ are subfields of K.

7. Let $f(x) = x^p - x - b \in \mathbb{F}_p[x]$ where $b \in \mathbb{F}_p$, $b \neq 0$. Show that $f(x)$ is irreducible over $\mathbb{F}_p$. What is the splitting field K? What is the Galois group $G = \mathrm{Gal}(K/\mathbb{F}_p)$ (both in terms of the action on K and as an abstract group)?
   Hint: It might be easiest to *start* by considering the action of G on a root α of $f(x)$.