University of Oklahoma Department of Mathematics
Real Analysis Qualifier Exam
January, 2025

Directions: Answer each question on a separate page (or pages), writing your name in the upper right corner of each page and the problem number in the upper left corner of each page.

Please turn in a page for each problem, even if you are leaving the problem blank.

Each problem is worth 10 points. Completely justify your work and state which theorems or results you are citing and how you have confirmed the hypotheses of those theorems. You have 3 hours to complete this exam.

Notation: The real line is denoted by $\mathbb{R}$, outer measure on subsets of $\mathbb{R}$ is denoted by $|\cdot|$, Lebesgue measure is denoted by λ, integration with respect to Lebesgue measure may be denoted with $\mathrm{d}\lambda$ or just $\mathrm{d}x$.

We will use $\mathcal{S}$ to indicate an arbitrary σ-algebra on a set X. In the case when $X \subseteq \mathbb{R}$, we will use $\mathcal{B}$ to indicate the Borel σ-algebra and $\mathcal{L}$ to indicate the Lebesgue σ-algebra.

1. Prove the "Approximation by Simple Functions" for Lebesgue measure on $\mathbb{R}$: Let f be a bounded measurable function on $\mathbb{R}$. For any $\epsilon > 0$, there exist simple functions ϕ and ψ such that

$$\phi \leq f \leq \psi \quad \text{and} \quad 0 \leq \psi - \phi < \epsilon \text{ on } \mathbb{R}.$$

2. Let $(X, \mathcal{S}, \mu)$ be a measure space and let f be in $L^1(\mu)$. Let $\{A_k\}_{k=1}^{\infty}$ be disjoint measurable subsets of X and let $A = \bigcup_{k=1}^{\infty} A_k$. Prove carefully that $\displaystyle\int_A f \, d\mu = \sum_{k=1}^{\infty} \int_{A_k} f \, d\mu$.

3. A real-valued function f on $\mathbb{R}$ is said to be a Lipschitz function if there exists a constant $K > 0$ such that for all $x, y \in \mathbb{R}$, $|f(x) - f(y)| \leq K|x - y|$.

   (a) Prove that a Lipschitz function $f : [a, b] \to \mathbb{R}$ maps sets of Lebesgue measure zero onto sets of Lebesgue measure zero.

   (b) Prove that a Lipschitz function on $\mathbb{R}$ maps F_σ sets to F_σ sets. (Recall that an F_σ set is a countable union of closed sets.)

   (c) Prove that a Lipschitz function on $\mathbb{R}$ maps Lebesgue measurable sets to Lebesgue measurable sets.

   (d) Provide an example showing that continuity is not enough to insure the property in part (c). (Hint: Let f be the Cantor ternary function and let $f_1(x) = f(x) + x$.)

4. Let $f : \mathbb{R} \to \mathbb{R}$.

   (a) Is either of the following statements stronger than the other? Please justify your answer.

   (i) f is continuous almost everywhere.

   (ii) f agrees with a continuous function almost everywhere.

   (b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is monotone. Is it necessarily true that:

   (i) f agrees almost everywhere with an everywhere differentiable function?

   (ii) f is almost everywhere differentiable?

   Please justify your answers.

5. Let μ be a finite measure on $(0, 1)$ such that μ and Lebesgue measure (denoted λ) are mutually singular. Show that, for every $\epsilon \in (0, \mu(0, 1))$, there exists a finite collection of disjoint open intervals (x_k, y_k), $k = 1, \ldots, n$, such that $\displaystyle\sum_{k=1}^{n} (y_k - x_k) < \epsilon$ and $\displaystyle\sum_{k=1}^{n} \mu(x_k, y_k) \geq \mu(0, 1) - \epsilon$.

6. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in $L^2(\mathbb{R})$.

   (a) Suppose that for all $g \in L^2(\mathbb{R})$ we have

$$\lim_{k \to \infty} \int_{\mathbb{R}} f_k g \, d\lambda = \int_{\mathbb{R}} f g \, d\lambda$$

   and $\|f_k\|_2 \to \|f\|_2$. Prove that $f_k \to f$ in $L^2(\mathbb{R})$.

   (b) Give an example to show that part (a) may fail if we do not assume $\|f_k\|_2 \to \|f\|_2$.