Jan 6, 2025 **TOPOLOGY QUALIFYING EXAM** 9:00-12:00pm

Instructions: You have three hours to complete the exam. There are three parts to the exam; follow the instructions for each part and clearly label the problems you are solving.

I

Definitions and examples. Complete all the problems below. When providing examples, you don't need to give a detailed proof, but make sure to give a clear description of each example.

1. Give the definition of a *basis* of a topological space X. (Warning: you are assuming that you already have a topology on X.) Describe two distinct bases for the standard (euclidean) topology on $\mathbb{R}^2$, and say how you would verify that these two bases generate the same topology on $\mathbb{R}^2$.
2. Define what it means for a topological space to be *locally-connected*. Give an example that shows that a connected space is not necessarily locally-connected.
3. Let $\{X_\alpha \mid \alpha \in J\}$ be an non-empty indexed family of non-empty topological spaces. Define the *product* and the *box* topology on $X = \prod_{\alpha\in J} X_\alpha$. Give a sufficient condition on when these two topologies are the same.
4. Define what it means for a map $f : X \to Y$ be a *homotopy equivalence*. Give an example of two spaces that are homotopy equivalent but not homeomorphic.
5. Define what it means for a subspace $A \subset X$ to be a *retract*. Give an example of $A \subset X$ which is not a retract.

II

Solve THREE of these problems. All proofs should be clear and concise. State clearly any theorems you are using.

1. Consider the set $\mathbb{Z}$ of integer numbers endowed with the cofinite topology (only finite sets and the entire set are closed).
   (a) Is this space compact? Prove your statement.
   (b) Is it connected? Prove your statement.
   (c) Is it T1 (points are closed)? Hausdorff? Regular? Normal? metrizable? Explain.
2. (a) Give a detailed proof that the continuous image of a connected space is connected.
   (b) Show that if a subspace $A \subset X$ is connected then so is its closure $\bar{A}$.
3. (a) Give a detailed proof that a compact subspace of a Hausdorff space is closed.
   (b) Show that a continuous open map from a compact space to a connected Hausdorff space is surjective.
4. (a) Define what it means to be a quotient map $p : X \to Y$.

(b) Which of the following properties are preserved by quotient maps: compactness, Hausdorff, discrete topology. (Preserved means that if X has the property then so does any quotient Y of X.) If a property is preserved, then give a quick proof. If not, then give a counter-example.

5. (a) Show that if X is locally-compact and Hausdorff then its one-point compactification $\bar{X}$ is Hausdorff.

   (b) Let X be an uncountable set with the discrete topology. Prove that the one-point compactification $\bar{X}$ of X cannot be embedded into the plane $\mathbb{R}^2$.

III

Solve THREE of these problems. All proofs should be clear and concise. State clearly any theorems you are using.

1. Suppose X is a connected CW complex that has just one cell in dimensions 0 and 2, but may have any number of cells in other dimensions. Also suppose the attaching map of the 2-cell $\varphi : S^1 \to X^1$ is NOT surjective, where X^1 denotes the 1-skeleton of X. Consider these questions about X:

   (a) Is X simply connected?

   (b) Is $\pi_1(X)$ infinite?

   (c) Is $\pi_1(X)$ abelian?

   For each one, if the question can be answered "yes" or "no" based solely on the information provided, give such an answer and justify it. Otherwise, give examples to show that the answer is not determined by the information provided.

2. Show that any null-homotopic map $f : S^1 \to S^1$ has to have a fixed point.

3. (a) State Van Kampen's theorem for $X = A \cup B$. (Only do it for two subsets.)

   (b) Give a detailed proof that $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$ using Van Kampen's theorem. You must write down the sets A and B explicitly.

4. (a) Define what it means for $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ to be a covering map, and state what you know about p_*. Define what it means for p to be a normal covering.

   (b) Compute the fundamental group of $\mathbb{RP}^2 \vee \mathbb{RP}^2$. (You don't have to give a detailed proof like you did for the previous problem.) Show that $\mathbb{RP}^2 \vee \mathbb{RP}^2$ does not admit a 3-sheeted connected normal covering.

5. (a) State the general lifting criterion for continuous maps $f : (Y, y_0) \to (X, b_0)$ into the base space of a covering map $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$. Make sure you also state the topological conditions on Y.

   (b) Let $q : (\tilde{Y}, \tilde{y}_0) \to (Y, y_0)$ be the universal cover, and $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ an arbitrary covering map. Show that for any continuous map $f : (Y, y_0) \to (X, x_0)$, there is a lift $\tilde{f} : (\tilde{Y}_0, \tilde{y}_0) \to (\tilde{X}_0, \tilde{x}_0)$ such that $p\tilde{f} = fq$.