Algebra Qualifying Exam – August 2025

Instructions:

i) There are two sections to the exam with 4 problems each. You are required to do any 3 of 4 problems from each section. If you attempt more than 3 problems from any section, your score will be based on your best three.

ii) Some of the problems are divided into parts. If you cannot complete a particular part, it is ok to use it for subsequent parts.

Section A

i) Suppose $0 < p < q$ are two odd primes and let G be a group with order $2pq$.

   a) Show that G is not simple.

   b) Show that G has a normal subgroup of order pq.

ii) Let G be a finite group such that the center $Z(G)$ of G has index n in G. Show that the size of each conjugacy class in G is at most n.

iii) Let D be an integral domain and assume that $D[x]$ is a principal ideal domain (PID).

   a) Let α be any non-zero element in D. Show that the ideal $I := \langle \alpha, x \rangle$, generated by α and x, satisfies $I = D[x]$.

   b) Show that every non-zero element of D has an inverse, and hence D is a field.

   The above problem proves a converse of "If F is a field then $F[x]$ is a PID"

iv) Is the ideal $I = \langle 1 + \sqrt{-3}, 1 - \sqrt{-3} \rangle \subset \mathbb{Z}[\sqrt{-3}]$ a principal ideal? Explain your answer.

Section B

i) Let $\alpha = \sqrt{3} + \sqrt{5}$.

   a) Determine the minimal polynomial of α over $\mathbb{Q}$.

   b) Show that $\sqrt[3]{5} \notin \mathbb{Q}(\alpha)$.

   c) Show that $\mathbb{Q}(\alpha)$ is Galois over $\mathbb{Q}$.

ii) Let K be a field extension of a field F with $1 < [K : F] < \infty$.

   a) Show that there is a F-module homomorphism $\phi : K \otimes_F K \to K$ such that $\phi(x \otimes y) = xy$.

   b) We know that $K \otimes_F K$ is a commutative ring with multiplication defined on generators by $(x \otimes y) \cdot (x' \otimes y') := xx' \otimes yy'$ and has the unit $1 \otimes 1$, where 1 is the unit in F (and K). Show that ϕ (from part (a) above) is a ring homomorphism and has a non-trivial kernel i.e. $1 \subsetneq \ker(\phi) \subsetneq K \otimes_F K$.

   c) Show that the ring $K \otimes_F K$ is not a field.

iii) Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree p, where p is an odd prime number. Let E be the splitting field of $f(x)$ over $\mathbb{Q}$ and $\alpha \in E$ be a zero of $f(x)$. Let $G = \mathrm{Gal}(E/\mathbb{Q})$ and $H = \mathrm{Gal}(E/\mathbb{Q}(\alpha))$. Suppose $H \neq \{1\}$.

   a) Show that $[G : H] = p$.

   b) Show that $\gcd(|H|, p) = 1$.

   c) Show that H is not a normal subgroup of G.

iv) Let p be an odd prime and let $\zeta_p := e^{\frac{2\pi i}{p}} = \cos(\frac{2\pi}{p}) + i \sin(\frac{2\pi}{p})$.

   a) Show that $[\mathbb{Q}(\zeta_p) : \mathbb{Q}(\cos(\frac{2\pi}{p}))] = 2$.

   b) Show that $\mathbb{Q}(\cos(\frac{2\pi}{p}))$ is a Galois extension of $\mathbb{Q}$ and the Galois group is cyclic of order $(p-1)/2$.