Analysis PhD Qualifying Exam   August 2025

In the problems below:

- All integrals are with respect to the Lebesgue measure and $L^p([0,1])$ is with respect to the Lebesgue measure on $[0,1]$.
- $C([0,1])$ denotes complex valued continuous functions on $[0,1]$ and $C([a,b],\mathbb{R})$ denotes real valued continuous functions on $[a,b]$. The algebra of polynomials with complex coefficients is denoted by $\mathbb{C}[x]$.

Please justify your answers!

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1. Let $f_n \in C([a,b],\mathbb{R})$ for each $n \in \mathbb{N}$. Does $F_n(x) := \int_a^x \sin(f_n(t))dt$ have a uniformly convergent subsequence?

2. Show that the following subsets of $(0,1) \subset \mathbb{R}$ are Borel sets and find their Lebesque measures: a) Points with multiple decimal expansions (such as $0.5 = 0.4999\ldots$).
b) Points that do **not** have a 7 in their decimal expansion.
c) Points that **have** both a 4 and a 7 in their decimal expansion.

3. Find the σ-algebras generated by **(i)** $\{\{r\} \mid r \in \mathbb{R}\}$ and **(ii)** $\{(q,\infty) \subset \mathbb{R} \mid q \in \mathbb{Q}\}$. Define a measure μ on each with $\mu(\mathbb{R}) = 1$.

4. Compute $\lim_{n\to\infty} \int_0^1 \frac{\pi n + \sin(nx)}{2n + \cos(n^2 x)} dx$.

5. Compute $\int_0^1 \int_y^1 x^{-3/2}\, cos(\frac{\pi y}{2x})\, dxdy$.

6. Prove or disprove: a) If $1 \leq p \leq q < \infty$ then $L^q([0,1]) \subseteq L^p([0,1])$.
b) If $1 \leq p < q < \infty$ then there is no containment between $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$.
c) $L^1([0,1]) = \bigcup_{p>1} L^p([0,1])$.
d) If $T : \mathbb{C}[x] \to \mathbb{C}$ is $T(f) := f'(0)$ for all $f \in \mathbb{C}[x]$ where $\mathbb{C}[x]$ is regarded as a subspace of $C([0,1])$ with $\|\,\|_\infty$ then T is a bounded linear map. Hint: Consider $1-(1-x)^n$.

7. Show that if X is an **incomplete** inner product space then there is a bounded linear functional on X which does not equal $\langle -, x\rangle$ for any $x \in X$.

8. Prove that if the measure ν is absolutely continuous with respect to the **finite** measure μ then ν is σ-finite.