Topology Qualifying Exam, August 2025

Instructions

Attempt **12 out of 18** problems. Choose **at least 1 problem from each section**. Clearly justify all your answers. Partial credit may be awarded for meaningful progress.

Part I: Topological spaces and continuous maps

Problem 1. State the definition of topology.

Problem 2. Let A be a subset of a topological space. Prove that A is closed if and only if it contains all its limit points.

Problem 3. Let X be a compact Hausdorff space and $f : X \to X$ a continuous bijection. Prove that f is a homeomorphism.

Part II: Connectedness and compactness

Problem 1. State the definition of connectedness and compactness.

Problem 2. Prove that the image of a compact space under a continuous map is compact.

Problem 3. Let $Y \subset X$ and let X and Y be connected. Show that if A and B form a separation of $X \setminus Y$, then $Y \cup A$ and $Y \cup B$ are connected.

Part III: Countability and separation axioms

Problem 1. State the definition of first-countable and second-countable.

Problem 2. Prove that every metric space is normal. Is the converse true? Justify your answer.

Problem 3. Prove that a subspace of a regular space is regular.

Part IV: Fundamental groups

Problem 1. Let X be a topological space. What are the elements of the fundamental group $\pi_1(X, x)$ of X?

Problem 2. State the definition of group operation on $\pi_1(X, x)$.

Problem 3. Show that $\mathbb{R}$ is contractible; namely, show that the identity map $i\colon \mathbb{R} \to \mathbb{R}$ is null-homotopic.

Part V: Covering spaces

Problem 1. Prove that the map $f\colon \mathbb{R} \to \mathbb{S}^1$ given by $f(x) = (\cos(2\pi x), \sin(2\pi x))$ is a covering map.

Problem 2. Prove that if $p\colon E \to B$ and $p'\colon E' \to B'$ are covering maps, then $p \times p'\colon E \times E' \to B \times B'$ is a covering map.

Problem 3. Prove that there is no continuous antipode-preserving map $g\colon S^2 \to S^1$. Recall that $f\colon S^n \to S^m$ is *antipode-preserving* if $f(x) = -f(-x)$ for any $x \in S^n$.

Part VI: van Kampen's Theorem

Problem 1. State the definition of homotopy equivalence.

Problem 2. State the Seifert-van Kampen theorem.

Problem 3. Use the Seifert-van Kampen theorem to compute the fundamental group of the theta space. In particular, write the theta space as a union of two path-connected open subsets A and B containing the base point with $A \cap B$ being path-connected.