Algebra Qualifying Exam – January 2026

Instructions:

i) There are two sections to the exam with 4 problems each. You are required to do any 3 of 4 problems from each section. If you attempt more than 3 problems from any section, your score will be based on your best three.

ii) Some of the problems are divided into parts. If you cannot complete a particular part, it is ok to use it for subsequent parts.

Section A

i) Let G be a finite group and let $Z(G)$ be its center, and let $\mathrm{Aut}(G)$ be its group of automorphisms.

a) Suppose $G/Z(G)$ is cyclic with generator $xZ(G)$. Show that, for any $g \in G$, there exists $m \in \mathbb{Z}$ and $z \in Z(G)$ such that $g = x^m z$. Hence, conclude that G is abelian.

b) Suppose $\mathrm{Aut}(G)$ is cyclic, then show that G is abelian.

c) Show that if G is abelian, and $|G| > 2$, then G has an automorphism of order 2.

d) Deduce that no finite group of order > 2 has a cyclic automorphism group of odd order.

ii) Let p be a prime and let G be a non-abelian group of order p^3.

a) If $Z(G)$ is the center of G, show that $|Z(G)| = p$.

b) Let $g \in G - Z(G)$ and let $C(g)$ be its conjugacy class in G. Show that $|C(g)| = p$.

c) Show that there are $p^2 + p - 1$ distinct conjugacy classes in G.

iii) Consider the ring $\mathbb{Z}[x]$.

a) Let $I = (2, x)$ be the ideal of $\mathbb{Z}[x]$ generated by 2 and x. Show that I is a prime ideal in $\mathbb{Z}[x]$.

b) Let P be a prime ideal in $\mathbb{Z}[x]$ such that $P \cap \mathbb{Z} \neq \{0\}$. Show that P is generated as an ideal in $\mathbb{Z}[x]$ by at most two elements.

iv) Let R be a ring, and define

$$N = \{x \in R : \text{ there exists } n \in \mathbb{Z} \text{ such that } x^n = 0\}.$$

a) If R is commutative, then show that N is an ideal in R.

b) Give an example of a non-commutative ring R such that N is not an ideal. Explain your answer.

Section B

i) Let $F \subset K$ be field extensions. Let $\alpha \in K$ be an algebraic element of odd degree over F. Show that $F(\alpha) = F(\alpha^2)$.

ii) Let $\alpha = \sqrt{2 + \sqrt{2}}$.

a) Find the minimal polynomial of α over $\mathbb{Q}$ and determine the degree of the splitting field of α over $\mathbb{Q}$.

b) Show that the Galois group of the splitting field is cyclic.

iii) Let V be the vector space of all polynomials of degree less than or equal to 3 with coefficients in $\mathbb{R}$. Let $T : V \to V$ be the linear transformation given by $T(f(x)) = f(x + 1)$.

a) Find the matrix for T with respect to the standard basis for V given by $\{1, x, x^2, x^3\}$.

b) Find the characteristic and minimal polynomial of T.

c) Give the Jordan canonical form for T.

iv) Let ζ be a primitive eighth root of unity and let $K = \mathbb{Q}(\zeta)$. Determine $\mathrm{Gal}(K/\mathbb{Q})$ and all the intermediate fields between $\mathbb{Q}$ and K.