Analysis PhD Qualifying Exam   January 2026

In the problems below all integrals are with respect to the Lebesgue measure and $L^p([0,1])$ and $L^p(\mathbb{R})$ are with respect to the Lebesgue measure.

Please justify your answers!

1. Let $f_n : [0,1] \longrightarrow \mathbb{R}$ be differentiable with $|f_n(0)| \leq 1$ for all $n \in \mathbb{N}$ and $|f_n'(t)| \leq 1$ for all $n \in \mathbb{N}$ and for all $t \in [0,1]$. Prove that there is a subsequence which converges uniformly to a continuous function $f : [0,1] \longrightarrow \mathbb{R}$.

2. Let (X,d) be a metric space. A function $f : X \to \mathbb{R}$ is called *lower semicontinuous* if for every $x \in X$ and every $\epsilon > 0$, there exists a $\delta > 0$, such that for all $y \in X$ with $d(y,x) < \delta$, we have $f(y) > f(x) - \epsilon$. Show that every lower semicontinuous function is Borel measurable.

3. Prove that if $E \subset \mathbb{R}$ is not Lebesgue measurable then we can find $\delta > 0$ such that if $A \subset E \subset B$ with A and B measurable then $\mu(B \setminus A) > \delta$.

4. Prove or disprove: There is a sequence $\{f_n\} \subset L^1(\mathbb{R})$ such that $\lim_{n\to\infty} \|f_n\|_1 = 0$ but $\limsup_{n\to\infty} f_n(x) = \infty$ for all $x \in \mathbb{R}$.

5. Show that if $f : [0,1] \longrightarrow \mathbb{C}$ is measurable then $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$.

6. Let $f : [0,1] \longrightarrow [0,\infty)$ be Lebesgue measurable and let $g_n = \dfrac{f^n}{1+f^n}$ for $n \in \mathbb{N}$.
Show that $\displaystyle\int_0^1 g_n$ and $\displaystyle\lim_{n\to\infty} \int_0^1 g_n$ both exist and are finite.

7. Prove or disprove: $L^\infty([0,1])$ is separable with respect to $\| \|_\infty$.

8. Let $f : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ be defined as:

$$f(x,y) = \begin{cases} 1 & \text{if } xy \in \mathbb{Q} \\ 0 & \text{if } xy \notin \mathbb{Q} \end{cases}$$

(a) Show that f is Lebesgue measurable.
(b) Compute $\displaystyle\int_0^1 \int_0^1 f(x,y)\, dx dy$.

9. Let A and B be bounded linear operators on the Hilbert space H and $x \in H$. Prove that there is a $y \in H$ such that $\langle Ax, Bz \rangle = \langle y, z \rangle$ for all $z \in H$.

10. Let Σ be a sub σ-algebra of Lebesgue measurable subsets of $[0,1]$. Prove that for all $f \in L^1([0,1])$ there is a Σ-measurable function g, such that

$$\int_A f\, d\mu = \int_A g\, d\mu$$

for all $A \in \Sigma$.