# Topology Qualifying Exam, January 2026

## Instructions

Attempt **12 out of 18** problems. Choose **at least 1 problem from each section**. Clearly justify all your answers. Partial credit may be awarded for meaningful progress.

## Part I: Topological spaces and continuous maps

**Problem 1.** State the definition of a continuous function.

**Problem 2.** Suppose that $f\colon X \to Y$ is a continuous function. If $x$ is a limit point of the subset $A$ of $X$, is $f(x)$ a limit point of $f(A)$? You need to either prove the statement or give a counter example.

**Problem 3.** Show that if $A$ is closed in $X$ and $B$ is closed in $Y$, then $A \times B$ is closed in $X \times Y$.

## Part II: Connectedness and compactness

**Problem 1.** State the definition of a separation of a topological space.

**Problem 2.** Prove that if $f\colon X \to Y$ is a continuous map from a compact space to a Hausdorff space, then $f$ is a closed map (i.e., the image of a closed set is closed).

**Problem 3.** Prove that a topological space $X$ is disconnected if and only if there is a continuous surjection of $X$ onto the discrete space $\{0, 1\}$.

## Part III: Countability and separation axioms

**Problem 1.** State the definition of a space having a countable basis at $x$.

**Problem 2.** Prove that every compact Hausdorff space is normal.

**Problem 3.** Prove that any countable product $\prod_{k=1}^{\infty} X_k$ of first countable spaces $X_k$ is first countable.

## Part IV: Fundamental groups

**Problem 1.** State the definition of two continuous maps $f\colon X \to Y$ and $g\colon X \to Y$ being homotopic.

**Problem 2.** Prove that being homotopic is an equivalence relation.

**Problem 3.** Recall that if $h\colon (X, x_0) \to (Y, y_0)$ is a continuous map, we define $h_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $h_*([f]) := [h \circ f]$. Prove that $h_*$ is well-defined and is a group homomorphism.

## Part V: Covering spaces

**Problem 1.** Prove that the map $p\colon \mathbb{R} \to S^1$ given by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is a covering map.

**Problem 2.** Prove that every connected cover of the circle is normal.

**Problem 3.** Let $E$ be path connected and let $p\colon E \to B$ be a covering map with $p(e_0) = b_0$. Prove that $p_*\colon \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective.

## Part VI: van Kampen's Theorem

**Problem 1.** State the Seifert-van Kampen theorem.

**Problem 2.** Use the Seifert-van Kampen theorem to compute the fundamental group of the 2-sphere $S^2$. In particular, write $S^2$ as a union of two path-connected open subsets $A$ and $B$ containing the base point with $A \cap B$ being path-connected.

**Problem 3.** Show that the complement of a finite set of points in $\mathbb{R}^3$ is simply-connected.