Qualifying Exam in Algebra, Fall 2026

This test has two sections. Solve 3 problems from each section. If you attempt more than 3 problems from any section, your score will be based on your best three. In your answers, indicate what theorems that you are using. Some of the problems are divided into parts. If you cannot complete a particular part, it is ok to use it for subsequent parts. All commutative rings are assumed to have a 1.

1. Part I

(1) Let p be a prime and let G be a group of order $p^a m$ where p does not divide m. Let P be a subgroup of G of order p^a, and let N be a normal subgroup of G of order $p^b n$ where p does not divide n.

(a) Prove that $|P \cap N| = p^b$ and $|PN/N| = p^{a-b}$.

(b) Show with an example that if N is not assumed to be normal that one can have $|P \cap N| = 1$ but $b \geq 1$.

(2) Let G be a group of order 56.

(a) Prove that G must have a normal Sylow subgroup.

(b) Deduce that G must be a semidirect product of two Sylow subgroups.

(c) Show that if the 2-Sylow subgroup is normal and isomorphic to $\mathbb{Z}/8\mathbb{Z}$, then $G \cong \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$.

(3) Let A and B be commutative rings. It is a fact (you don't need to prove) that $A \otimes_{\mathbb{Z}} B$ is a commutative ring where multiplication is given by $a \otimes b \cdot a' \otimes b' = (aa') \otimes (bb')$ on elementary tensors.

(a) Show that $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C}$ as rings. ($\cong$ means isomorphic here)

(b) Is it possible that A and B are not integral domains, but $A \otimes_{\mathbb{Z}} B$ is an integral domain? If yes, give an example and prove it's an example. If no, prove it's impossible.

(4) (a) A commutative ring R is called a local ring if it has a unique maximal ideal. Prove that if R is a local ring with maximal ideal M then every element of $R \setminus M$ is a unit. Prove conversely

that if R is a commutative ring in which the set of nonunits forms an ideal M, then R is a local ring with unique maximal ideal M.

(b) Show that the subring of $\mathbb{Q}$ consisting of elements expressible with odd denominators is a local ring with maximal ideal generated by 2.

2. Part II

(1) (a) Suppose T is a linear transformation on a vector space V over $\mathbb{Q}$. Suppose that the minimal polynomial of T is $(x^2+x-6)(x+4)$ and the characteristic polynomial is $(x^2+x-6)^2(x+4)$. Determine the rational canonical form of T.

(b) Is the rational canonical form of a linear transformation on a finite vector space always determined by the combination of its minimal and characteristic polynomials? I.e. can two matrices with distinct rational canonical forms have the same minimal and characteristic polynomials? Prove your answer.

(2) Consider the field extension $\mathbb{Q}(\alpha)$ of $\mathbb{Q}$ where α is a root of the polynomial x^4-2.
(a) Find $[Q(\alpha):\mathbb{Q}]$.

(b) Show that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is not a Galois extension.

(c) Find a subfield K of $\mathbb{Q}(\alpha)$ such that $\mathbb{Q}(\alpha)/K$ and $K/\mathbb{Q}$ are both Galois extensions. Prove that the extensions are Galois.

(3) Let K denote the splitting field of x^5-2 over $\mathbb{Q}$.

(a) Determine K. I.e. express it as $\mathbb{Q}(\alpha_1,\alpha_2,\ldots,\alpha_n)$ for explicit α_i.

(b) Describe the elements of Gal$(K/\mathbb{Q})$. Describe the automorphisms in terms of the actions on the generators α_i and prove that they are in fact automorphisms.

(4) Let R be a commutative ring. For an ideal I of R and an R-module M, recall

$$IM = \{\sum_{i=1}^{k} a_k m_k \mid a_i \in I, m_i \in M, k \geq 0\}.$$

(a) Prove that $R^n/IR^n \cong (R/I)^n$ where $\cong$ is isomorphism as R-modules.

(b) Prove that $R^n \cong R^m$ if and only if $m = n$. (Hint: Use part a and an appropriate ideal.)