ANALYSIS QUALIFYING EXAM

August, 2026

Instructions

1. You are not allowed to use any notes or aids for this test.
2. Write answers to the following questions on blank sheets of paper. Carefully order the pages as they are intended to be read.
3. At the end of the test, check that your name and ID are written on the cover page. Then number and initial all the pages.

Notations

1. All L^p spaces on subsets of $\mathbb{R}$ are with respect to the Lebesgue measure.
2. m denotes the Lebesgue measure.
3. $B_r(x)$ is the open ball centered at x with radius r.
4. For a given function f on an interval in $\mathbb{R}$, f' denotes the derivative of f.
5. $\operatorname*{Var}_{[0,1]} f$ denotes the total variation of f on $[0, 1]$.
6. For a complex number z, $\overline{z}$ denotes its complex conjugate.

Questions

1. Show that $\mathcal{B}_{\mathbb{R}^2}$, the Borel σ-algebra on $\mathbb{R}^2$, is generated by each of the following classes.
   (a) The class of all open half planes. An open half plane is a connected component in the complement of a line.
   (b) The class of all open discs of rational radii centered at points with rational coordinates.
2. Let μ be a Borel measure on $\mathbb{R}^2$. Define its maximal function
$$M\mu : \mathbb{R}^2 \to [0, \infty]$$
by
$$(M\mu)(x) = \sup_{r>0} \frac{\mu(B_r(x))}{m(B_r(x))}.$$
Show that $M\mu$ is Borel measurable.
3. Let $f : [0, 1] \to \mathbb{C}$ be Lebesgue measurable such that $|f| \le 1$. Show that for every $\epsilon > 0$, there exists a step function $g(x) = \sum_{k=1}^{N} a_k \chi_{I_k}$, where $a_k \in \mathbb{C}$ and each $I_k \subset [0, 1]$ is an interval, such that
$$m(\{x : |f(x) - g(x)| > \epsilon\}) < \epsilon.$$
4. Let $\{f_n\}_{n=1}^{\infty}$, f be measurable functions on X. Suppose $f_n \to f$ almost everywhere, and for every $\epsilon > 0$, there exists a finite measure set E such that $\|f_n \chi_{X\setminus E}\|_{L^1} < \epsilon$ for all n. Show that $f_n \to f$ in measure.
5. If $f(t)e^{-s_0 t} \in L^1(0, \infty)$ for some $s_0 \in \mathbb{R}$, we define the Laplace transform of f by
$$\mathcal{L}f : [s_0, \infty) \to \mathbb{C},$$
$$(\mathcal{L}f)(s) = \int_0^{\infty} f(t)e^{-st}\, dt.$$
Suppose $f(t)e^{-s_0 t} \in L^1(0, \infty)$.
   (a) Prove that $\mathcal{L}f$ is continuous on $[s_0, \infty)$, and $\lim_{s\to\infty} (\mathcal{L}f)(s) = 0$.
   (b) Prove that $\mathcal{L}f$ is differentiable on (s_0, ∞), and $(\mathcal{L}f)'(s) = -\mathcal{L}(tf)(s)$. Here tf means the function $tf(t)$.
6. Let μ, ν be two finite measures on $(X, \mathscr{M})$. They are called equivalent if $\mu \ll \nu$ and $\nu \ll \mu$. Prove that μ and ν are equivalent if and only if there exists $f \in L^1(X, \mu)$ such that $f > 0$ μ-a.e. and $d\nu = f\, d\mu$.
7. For $f \in \mathrm{BV}([0, 1])$, let $\|f\| = \operatorname{Var}_{[0,1]} f + |f(0)|$. Show that $\|\cdot\|$ is a norm on $\mathrm{BV}([0, 1])$, and that $\mathrm{BV}([0, 1])$ is a Banach space under this norm.
8. Let H be the set of absolutely continuous functions f on $[0, 1]$ such that
$$f(0) = f(1) = 0, \text{ and } \int_0^1 |f'(x)|^2\, dx < \infty.$$

(a) Show that $\langle f, g\rangle = \int_0^1 f'(x)\overline{g'(x)}\, dx$ is an inner product on H which makes it a Hilbert space.

(b) Show that for any $h \in L^2([0,1])$ there exists a unique $g \in H$ such that

$$\int_0^1 f'(x)\overline{g'(x)}\, dx = \int_0^1 f(x)h(x)\, dx$$

for all $f \in H$.