Part 1: Definitions and examples. Complete all four problems below. When providing examples, give a brief indication of why the desired properties hold; a detailed proof is not required.

1. (a) Let X be a space with topology $\mathscr{T}$. Say what it means for a collection $\mathscr{B}$ of subsets of X to be a *basis* for $\mathscr{T}$.

   (b) Give two different bases for the plane $\mathbb{R}^2$ (with its standard Euclidean topology). Say how you would verify that these two bases generate the same topology.

2. Let X be a topological space and $\sim$ an equivalence relation on X.

   (a) Explain what $X/\!\sim$ is (as a set) and define the quotient topology on it.

   (b) If Y is another space and $f\colon (X/\!\sim) \to Y$ is a function, give the criterion which characterizes when f is continuous, in terms of X.

   (c) Let $\sim$ be the equivalence relation on $X = S^1 \times [0,1]$ defined by $(\theta, t) \sim (\theta', t')$ if and only if $t = t' = 0$. The resulting quotient space is called the *cone* on S^1. Find a continuous bijection from $X/\sim$ to the unit disk $D^2 \subset \mathbb{R}^2$. Is it a homeomorphism? Explain.

3. Let $f\colon X \to Y$ be a continuous map and $x_0 \in X$ a basepoint.

   (a) Define the *induced homomorphism* f_* between appropriate fundamental groups.

   (b) Give examples where f_* is injective, and where f_* is not injective.

4. Give the definition of a *homotopy equivalence* from X to Y, and say what it means for X and Y to be *homotopy equivalent*. Give an example of two spaces that are homotopy equivalent but not homeomorphic.

Part 2: Point set topology. Solve **three** of these problems. All proofs should be clear and concise. State clearly and fully any theorems you are using.

5. Recall that in the *finite complement topology* on X, a set U is open if and only if $U = \emptyset$ or $X - U$ is finite.

   Now let X be an infinite set and let $\mathscr{B}$ be the collection of all sets $B \subseteq X$ such that $X - B$ has a finite and even number of elements.

   (a) Show that $\mathscr{B}$ is a basis for the finite complement topology on X.

   (b) Is the conclusion of part (a) true if X is finite? Explain.

6. Let $X = \mathbb{R} \times \mathbb{R}$ with the dictionary order topology. Give a detailed proof that X is not path connected. What are the path components of X?

7. Recall that $\mathbb{R}^\omega = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots$ is the set of sequences of real numbers. Let $A \subset \mathbb{R}^\omega$ be defined by

$$A = \{(x_i) \in \mathbb{R}^\omega \mid x_i = 0 \text{ for all but finitely many } i\}.$$

   (a) Prove that $\overline{A} = \mathbb{R}^\omega$ in the product topology.

   (b) Prove that $\overline{A} \neq \mathbb{R}^\omega$ in the box topology.

8. Let (X, d) be a compact metric space and suppose that $f\colon X \to X$ is an *isometry*. This means that

$$d(f(x), f(y)) = d(x, y) \text{ for all } x, y \in X.$$

   Prove that f is a homeomorphism. (The tricky part is surjectivity. Hint: if not surjective, consider *iterating* f.)

9. Let X be a non-compact, locally compact Hausdorff space.

   (a) Say what it means for X to be *locally compact*.

   (b) Define the one-point compactification $\widehat{X}$ of X. Give the topology of $\widehat{X}$ by saying what all of its open sets are.

   (c) Prove that $\widehat{X}$ is compact.

   (d) Find the one-point compactification of $S^1 \times (0, 1]$. Explain. (It is a familiar space.)

Part 3: Algebraic topology. Solve **three** of these problems. All proofs should be clear and concise. State clearly and fully any theorems you are using.

10. (a) State carefully the Van Kampen theorem (the version with two subsets).

    (b) Compute the fundamental group of the space X obtained from the torus $S^1 \times S^1$ by attaching a 2–cell along the circle $\{x_0\} \times S^1$.

11. Prove that the 2-sphere S^2 is not homeomorphic to the 3-sphere S^3.

12. State carefully the path lifting property of covering spaces (existence and uniqueness). Then, prove that if $p\colon \widetilde{X} \to X$ is a covering space with $\widetilde{X}$ and X path connected and $h\colon \widetilde{X} \to \widetilde{X}$ is a covering transformation, and $h(x) = x$ for some $x \in \widetilde{X}$, then h is the identity.

13. Describe all connected 3–sheeted covering spaces of the figure-eight space $S^1 \vee S^1$. Determine which of them are regular (also called normal) covering spaces.

14. For each pair of spaces (X, A), either give a retraction from X to A, or prove that a retraction does not exist:

    (a) $A =$ one point inside $X = \mathbb{R}^2$

    (b) $A =$ two points inside $X = \mathbb{R}^2$

    (c) X is the *theta graph*, having two vertices u, v and three edges a, b, c, each going from u to v; A is the subgraph with edges a and b (and vertices u, v)

    (d) A is a torus $S^1 \times S^1$ embedded as a subspace of $X = \mathbb{R}^3$
