What is an Euler characteristic?

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Norman Math Circle at the University of Oklahoma

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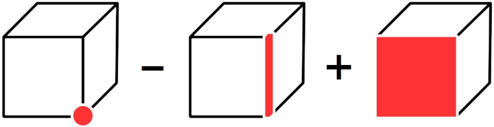

First Example

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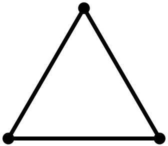

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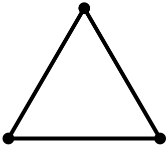

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- **Answer**: 3

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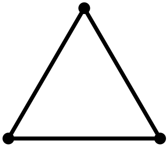

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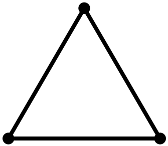

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First Example (continued)

Polygon	Vertices	Edges
Triangle	3	3

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What do we notice?

First Example (continued)

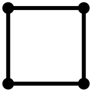
Polygon	Vertices	Edges
Triangle	3	3

What do we notice?

The number of vertices equals the number of edges.

Squares

Let's add a row to our table for squares:



Polygon	**Vertices**	**Edges**
Triangle	3	3
Square		

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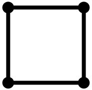

Polygon	Vertices	Edges
Triangle	3	3
Square	4	4

Again, the number of vertices equals the number of edges.

More polygons

This pattern continues:

Polygon	Vertices	Edges
Triangle	3	3
Square	4	4
Pentagon	5	5
Hexagon	6	6
⋮	⋮	⋮
n-gon	n	n

Polygons

Observation: For polygons, the number of vertices always equals the number of edges.

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- Let $V =$ the number of vertices.
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- So $V = E$ for polygons.

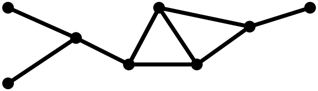
Another way to think of this:

- Let $\chi = V - E$.
- Then $\chi = 0$.
- χ is called the **Euler characteristic**.

Graphs

A set of vertices and edges is called a **graph**.

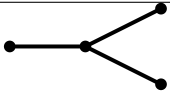
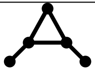
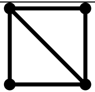
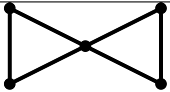
For example:

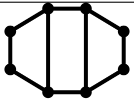
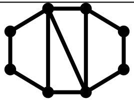
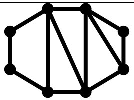
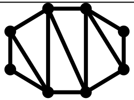


What is χ for this example?

Other graphs

Let's compute $\chi = V - E$ for some more graphs.

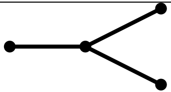
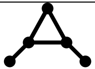
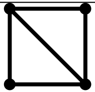
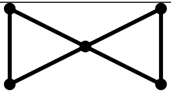
Shape	V	$-\ E$	$= \chi$
			
			
			
			
			
			

Shape	V	$-\ E$	$= \chi$
			
			
			
			

Do you notice any patterns?

Genus

The number of "holes" in a shape is called the **genus**, denoted g.

Shape	χ	g
	1	
	1	
	1	
	0	
	-1	
	-1	

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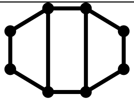
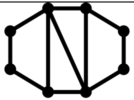
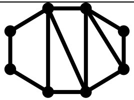
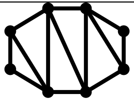
	-2	
	-3	
	-4	
	-5	

General formula for graphs

Do you see the pattern now?

g	χ
0	1
1	0
2	-1
3	-2
4	-3
5	-4
6	-5
$\vdots$	$\vdots$
g	

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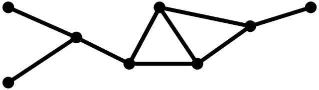
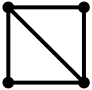
g	

The Euler characteristic χ of a genus g graph is:

Homotopy equivalence

Notice: If you can continuously deform one graph into another, then the two graphs have the same Euler characteristic χ.





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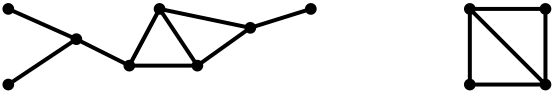

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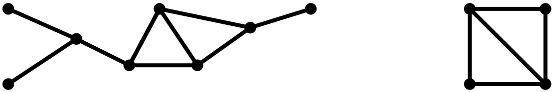

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- **Why** do homotopy equivalent graphs have the same Euler characteristic?

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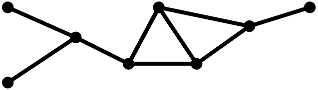
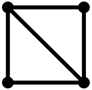




- Spaces related by continuous deformation like this are said to be **homotopy equivalent**.
- **Why** do homotopy equivalent graphs have the same Euler characteristic?
- **One possible answer:** If you shrink an edge down until it disappears, you get one fewer edge AND one fewer vertex, so $V - E$ stays the same.

Donuts and Coffee Mugs

Example: Donuts and coffee mugs are homotopy equivalent.

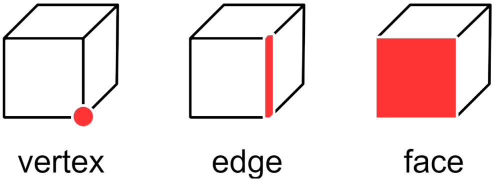
Surfaces

For surfaces, we define

$$\chi = V - E + F$$

where

- V is the number of vertices;
- E is the number of edges;
- F is the number of faces.



Euler Characteristics of Surfaces

Let's compute $\chi = V - E + F$ for a bunch of examples.

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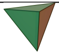
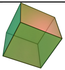
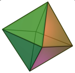
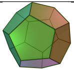
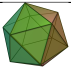
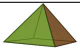
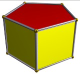
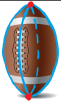
Tetrahedron					
Cube					
Octahedron					
Dodecahedron					
Icosahedron					

Name	Image	V	E	F	χ
Pyramid					
Prism					
Soccer Ball					
Football					
Basketball					

What do you notice about these Euler characteristics?

Euler Characteristics of Surfaces

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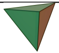
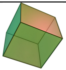
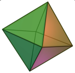
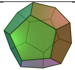
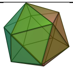
Tetrahedron		4	4	6	2
Cube		8	12	6	2
Octahedron		6	12	8	2
Dodecahedron		20	30	12	2
Icosahedron		12	30	20	2

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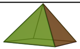
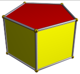
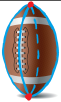
Pyramid		5	8	5	2
Prism		10	15	7	2
Soccer Ball		60	90	32	2
Football		2	4	4	2
Basketball		6	12	8	2

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Euler characteristic of a sphere

- Some shapes on the previous slide were curved. Notice that this doesn't really matter.

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Euler characteristic of a sphere

- Some shapes on the previous slide were curved. Notice that this doesn't really matter.
- All of the shapes on the previous slide can be deformed to look like spheres with a bunch of vertices and curvy edges drawn on them. This doesn't change anything.
- We might say $\chi(\text{sphere}) = 2$, since we get 2 for χ no matter how we draw the vertices/edges/faces on the sphere.

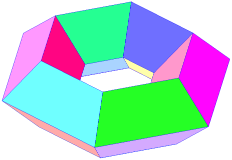
The torus

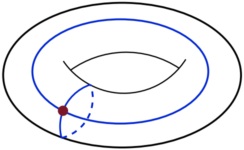
- For graphs, we saw that χ depends on the genus g.
- All the examples on the previous slide have $g = 0$ and $\chi = 2$.

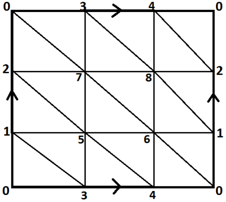
The torus

- For graphs, we saw that χ depends on the genus g.
- All the examples on the previous slide have $g = 0$ and $\chi = 2$.
- A torus (donut-shape) has genus 1.
- Do we get something different for χ?

Euler characteristic of the torus







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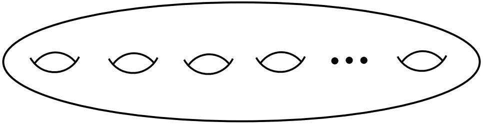
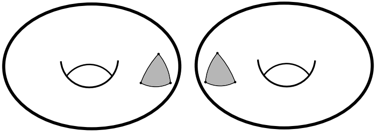
- Genus g?



Connect Sums

- Say we have two surfaces A and B.
- Let's assume each has a triangle face.
- What happens to the Euler characteristic if we remove a triangle face from each while keeping the vertices and edges the same?



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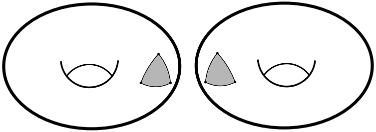

- A and B each lose a face, so $\chi = V - E + F$ goes down by 1 for each surface.

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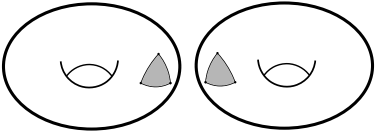
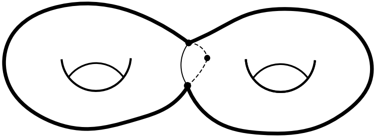


- Now let's glue the boundaries of the two triangles together to make a new surface called $A\#B$



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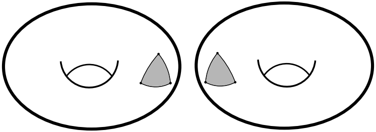

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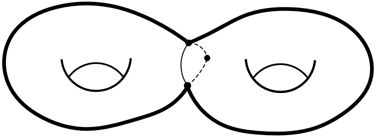
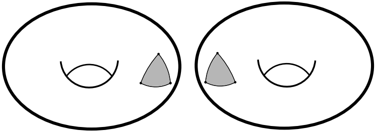


- When gluing, the total number of vertices goes down by 3. So does the total number of edges. So χ stays the same.

Connect Sums



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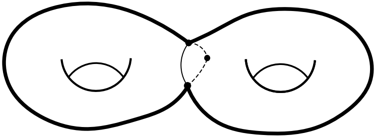

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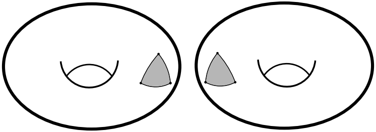

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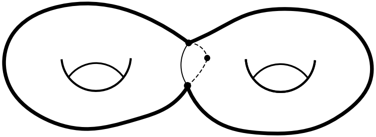
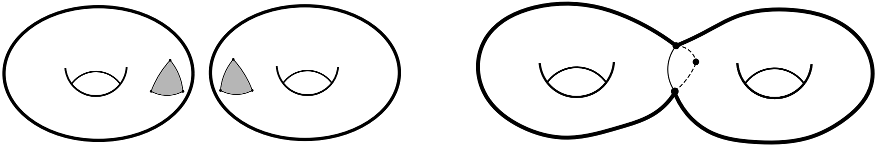


- When gluing, the total number of vertices goes down by 3. So does the total number of edges. So χ stays the same.
- So $\chi(A \# B) = \chi(A) + \chi(B) - 2$.

Connect sum with a torus



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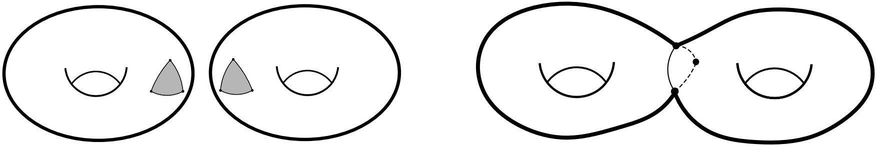

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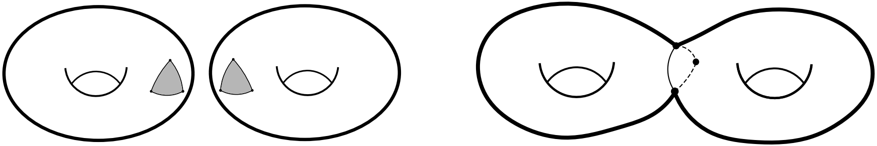

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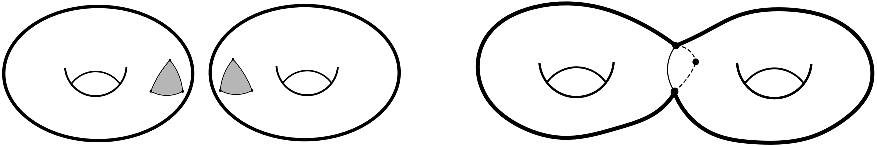

- $\chi(A\#B) = \chi(A) + \chi(B) - 2$.
- Suppose B is a torus, so $\chi(B) = 0$.
- Then $\chi(A\#B) = \chi(A) - 2$.
- Also, the genus of $A\#B$ is one more than the genus of A.
- So when we increase the genus by 1, it decreases χ by 2.

g	0	1	2	3	4	5	...	g
χ	2	0					...	

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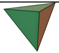
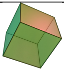
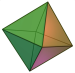
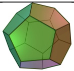
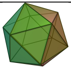
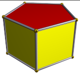
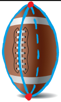
  - This tells you a bit about what the shape is.
  - This can be used, e.g., to identify abnormalities in medical imaging.

Image from `https://www2.math.upenn.edu/~ghrist/preprints/barcodes.pdf`

Thank you!

Name	Image	V	E	F	χ
Tetrahedron		4	4	6	2
Cube		8	12	6	2
Octahedron		6	12	8	2
Dodecahedron		20	30	12	2
Icosahedron		12	30	20	2

Name	Image	V	E	F	χ
Pyramid		5	8	5	2
Prism		10	15	7	2
Soccer Ball		60	90	32	2
Football		2	4	4	2
Basketball		6	12	8	2