Compass and Straightedge Constructions Worksheet

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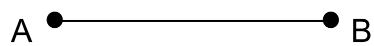
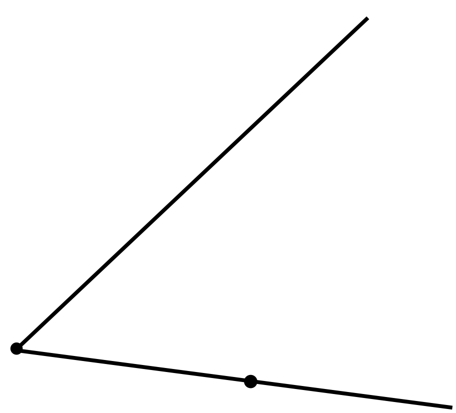
Problem 1. Using only a compass and straightedge, construct an equilateral triangle with a given line segment as a side.

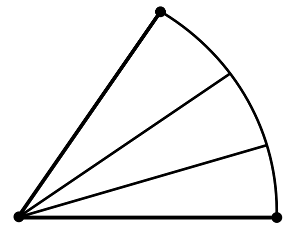


Problem 2. Given an angle, show how to bisect the angle with only a compass and straightedge.



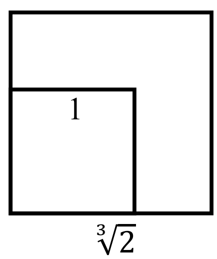
<u>Classical Questions</u>:

Trisecting an Angle: Can we trisect any given angle?



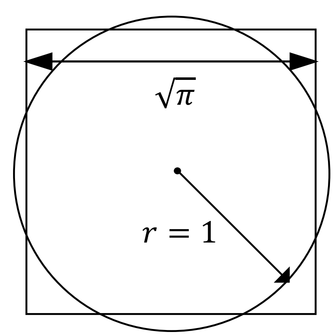
Answer: No

Doubling a Cube: Given the base of a cube of volume V, is it possible to construct the base of a cube of volume 2V?



Answer: No

Squaring the Circle: Given a circle of area A, is it possible to construct a square of area A?



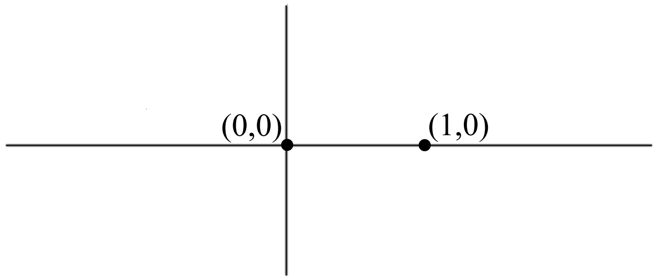
Answer: No

Regular polygons: Is it possible to construct, say, a regular 17-gon? What about a regular 18-gon?

Answer: Yes for the 17-gon, no for the 18-gon.

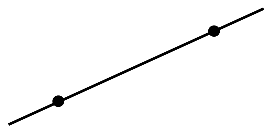
What is a constructible point?

(i) Start with $(0,0)$ and $(1,0)$. Call these points **constructible**.

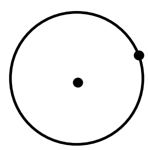


(ii) Given two constructible points, we may draw:

(a) A line through them:

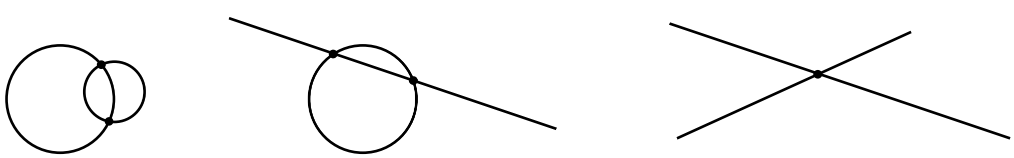


(b) A circle with center at one point and passing through the other.



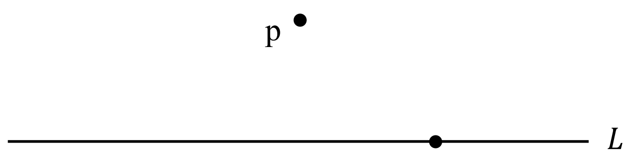
Such lines and circles are **constructible**.

(iii) The points of intersection of constructible lines and circles are also **constructible**.



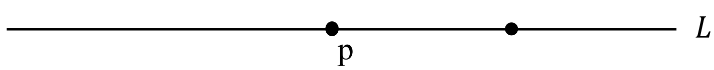
Problem 3. Given a line L and a point p, show that you can construct a line through p and perpendicular to L.

Case 1: p is not in L.



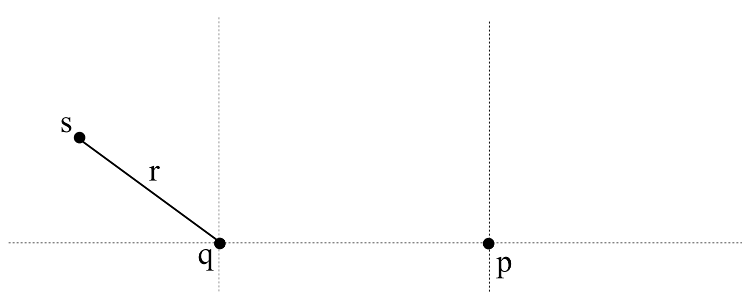
Case 2: p is in L.

Actually, you can just note that this is a special case of Problem 2 (why?).



Problem 4. Given two points q and r of distance d apart and a third point p, show that you can construct a circle of radius d centered at p.

Hint: Draw a line through p and q. Then use the previous problem to draw perpendiculars as below. **You don't have to do the full constructions.**



Definition:
We call a number d **constructible** if $|d|$ is the distance between 2 constructible points.

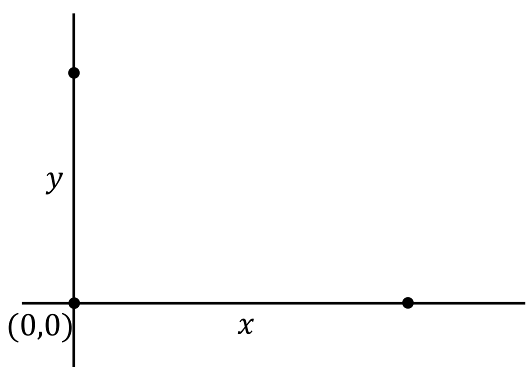
Example: Problem 4 says that if d is a constructible number, then we can construct a circle of radius |d| centered at any constructible point.

Example: If x is constructible, then so is $-x$ because $|x| = |-x|$.

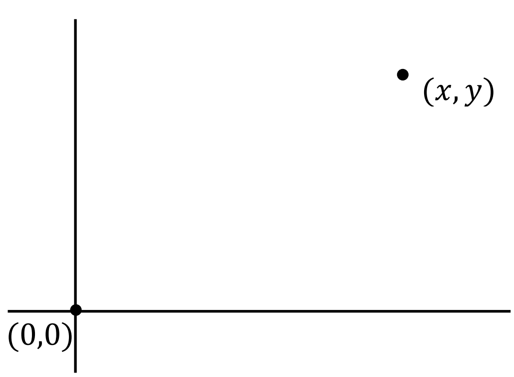
Problem 5. Show that a point (x, y) is constructible if and only if x and y are constructible numbers.

Hint: Use Problem 3. You don't have to do the whole construction to prove that it's possible.

If x and y are constructible, show that (x, y) is constructible.



If (x, y) is constructible, show that x and y are constructible.



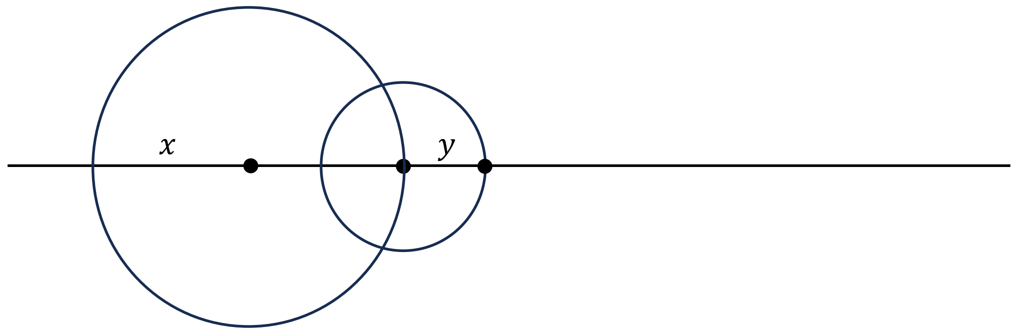
Alternative definition of a constructible number: A coordinate of a constructible point.

Problem 6: If x and y are constructible numbers, then so are:

(a) $x+y$
(b) $x-y$
(c) xy
(d) $\frac{x}{y}$ (assuming $y \neq 0$).

Solution:

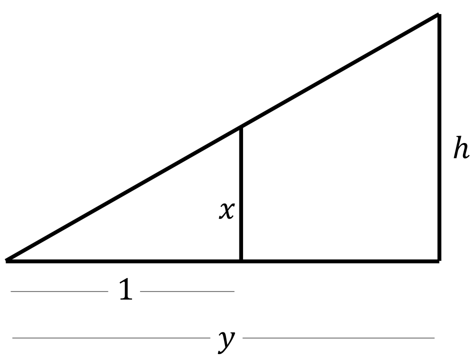
(a)



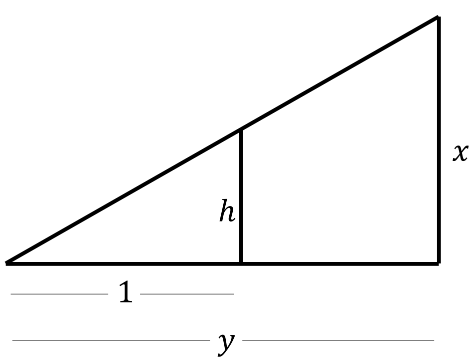
(b) This is similar to (a). After all, $x-y=x+(-y)$.

(c) If x and y are constructible, we can construct the similar right triangles as below.

What is h?



(d) Same as (c), but switch the roles of x and h.

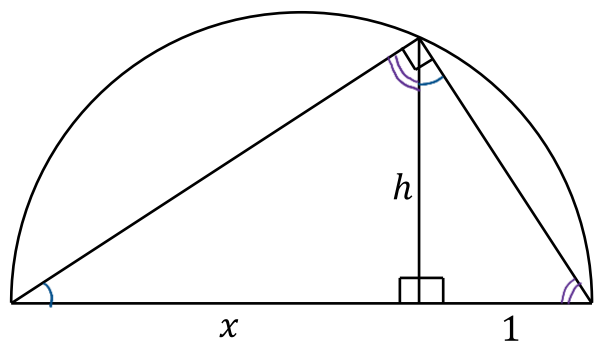


Problem 6 shows that the constructible numbers form what mathematicians call a **field**.

Note: It follows that every integer $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$ is constructible. Also, every rational number is constructible.

Problem 7. Prove that if x is a positive constructible number, then so is $\sqrt{x}$.

HINT: In the figure below, use similar triangles to express h in terms of x.



The left and right triangles are similar, so $\frac{h}{x} = \frac{1}{h}$

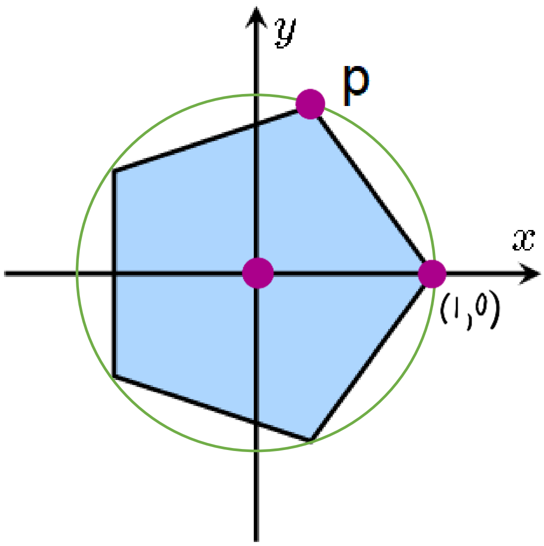
So $h^2 = x$, hence $h = \sqrt{x}$.

This proves the following theorem!

<u>Theorem</u>: Any point whose coordinates can be written using repeated sums, differences, products, quotients, and square roots of rational numbers is constructible.

Problem 8. Is the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ constructible?

Problem 9. Is a regular pentagon constructible?

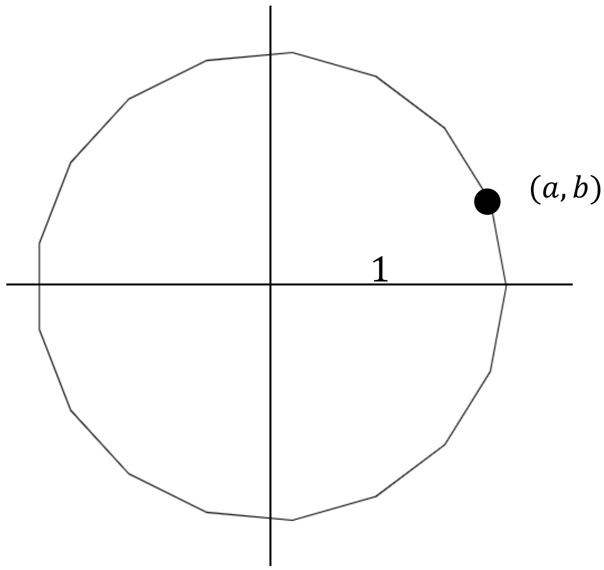


Using some trigonometry, it is possible (though tricky) to show that the coordinates for p are

$$p = \left(\frac{\sqrt{5}-1}{4}, \frac{\sqrt{2(\sqrt{5}+5)}}{4} \right)$$

Problem 10.

Is the regular 17-gon constructible?



In 1796, over 2000 years after the ancient Greeks had studied constructions, 19-year old Gauss found that:

$$a = \frac{-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} + 2\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}}{16}$$

So is a constructible?

Is the 17-gon constructible? (Hint: $b = \sqrt{1 - a^2}$)

The reverse is also true: ○

Theorem: Constructible numbers are **exactly** the ones which can be expressed using only repeated sums, differences, products, quotients, and square roots of rational numbers.

Sketch of Proof:
The idea is to consider the equations for constructible lines and circles. The coefficients for these equations are constructible numbers, and intersection points can be found using the quadratic formula. This expresses solutions in terms of sums, differences, products, quotients, and square roots of the coefficients.

For example, if we want to know when

$$y - y_0 = m(x - x_0) \quad \text{intersects} \quad (x - x_1)^2 + (y - y_1)^2 = r^2,$$

we plug

$$y = m(x - x_0) + y_0$$

in for y in the second equation and then solve for x. □

Using some pretty advanced ideas from abstract algebra (taught in a college or graduate level course), the above theorem implies the following:

Theorem: If c is a constructible number, then there is some nonzero polynomial $f(x)$ with integer coefficients such that $f(c) = 0$. Furthermore, the lowest possible degree for $f(x)$ is 2^k for some integer $k \geq 0$.

The minimal-degree polynomial $f(x)$ from the theorem is called the **minimal polynomial** of c.

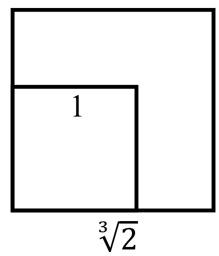
Example: π is not a constructible number because it is transcendental, meaning it is not a root of any polynomial with rational coefficients.

Example: 2 is constructible. Its minimal polynomial is $x - 2$.

Example: $\sqrt{2}$ is constructible. Its minimal polynomial is $x^2 - 2$.

Example: However, $\sqrt[3]{2}$ is not constructible, because its minimal polynomial is $x^3 - 2$.

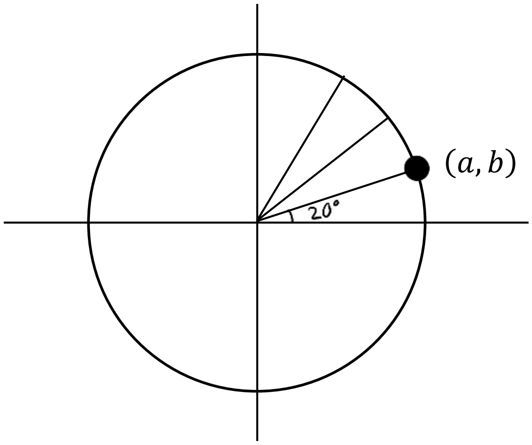
Problem 11. We just saw that $\sqrt[3]{2}$ is not constructible. Do you see why this shows that doubling a cube is impossible?



Problem 12. Why is squaring a circle impossible? That is, given a circle of radius 1, why can't you construct a square with the same area as the circle?

Answer: $\sqrt{\pi}$ is not constructible.

Problem 13. You can't trisect a $60°$ angle.



Plug $\theta = 20°$ into the identity:

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

Use $\cos 60° = 1/2$ and double both sides to get

$$1 = 8\cos^3 20° - 6\cos 20°$$

Let $a = \cos 20°$. Then a satisfies the equation

$$8a^3 - 6a - 1 = 0,$$

so has minimal polynomial of degree 3.

So a is not constructible.

Problem 14. In Problem 14, we showed that a 20° angle is not constructible. Why does this imply that a regular 18-gon is not constructible?