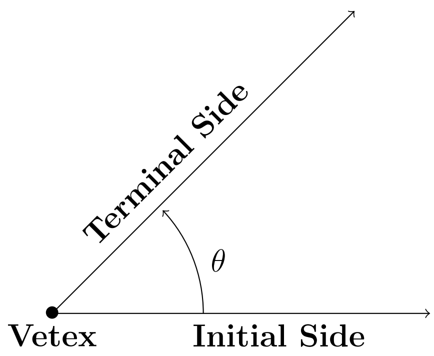
1 Right Triangle Trigonometry

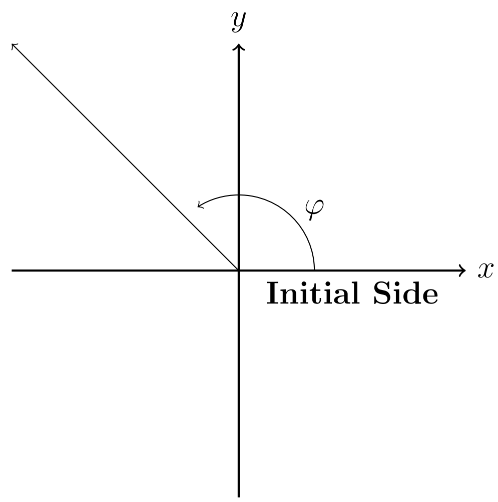
1.1 Angles and Angle Measure

1.1.1 Angles

Given a ray, an *angle* is determined by rotating the ray about its *vertex*, i.e. the endpoint of the ray. The starting position of the ray is called the *initial side* and the ending position is called the *terminal side*. Clockwise rotations generate *negative angles* whilst counterclockwise (ccw) angles generate *positive angles*.

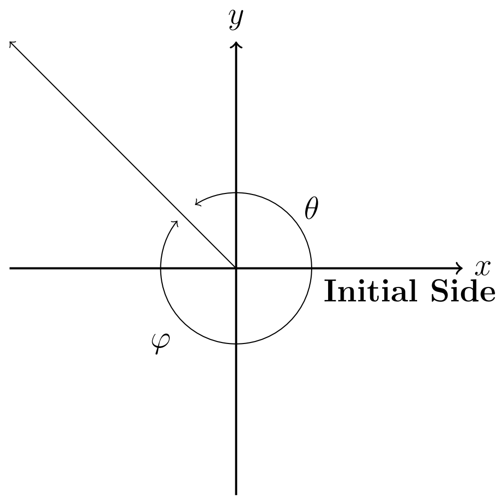


Definition 1.1. If the ray is placed in a coordinate system such that the initial side lies on the x-axis and the vertex is placed at the origin, such a ray is said to be in *standard position*.



Definition 1.2. Angles that have the same initial side and terminal sides are called *coterminal angles*.

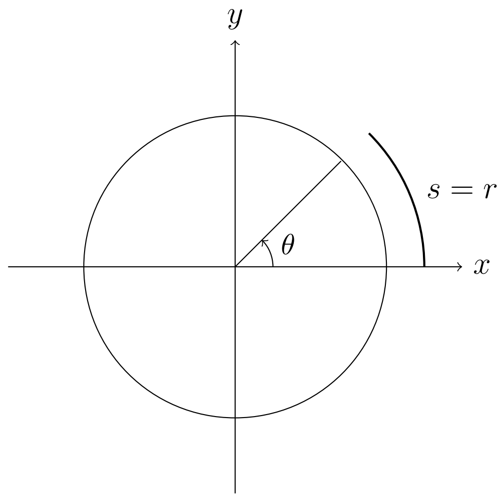
Angles θ and φ below are coterminal angles.



1.1.2 Radian Measure

Definition 1.3. A *central angle* of a circle is one whose vertex is the center of the circle.

Definition 1.4. One *radian* is the measure of a central angle θ that intersects an arc s equal in length to the radius r of the circle. More generally, the magnitude in radians of such an angle θ is defined to be the ratio of the length of the arc intercepted by θ to the radius of the circle, i.e. $\theta = \frac{s}{r}$ where s is the arc length and r is the radius of the circle.



From the definition of a radian, we obtain the arc length formula.

Arc Length Formula
For a circle of radius r, a central angle θ intercepts an arc of length s given by

$$s = r\theta$$

where θ is measured in radians.

Observe that if $s = 2\pi r$ (i.e. the full circumference of the circle), then $\theta = \frac{2\pi r}{r} = 2\pi$ (radians). In particular, there are 2π radians in a full circle.

Example 1

What is the length of the arc that corresponds with the angle whose measure is $\frac{2\pi}{3}$ radians in a circle of radius 3?

Solution: Since θ is already given in radians we simply substitute in our values $r = 3$ and $\theta = \frac{2\pi}{3}$ into the arc length formula to get $s = 3 \cdot \frac{2\pi}{3} = 2\pi$.

1.1.3 Degree Measure

The second (and often more intuitive) way to measure angles is in *degrees*, denoted by $^\circ$.

Definition 1.5. We define 1° to be $1/360$ of a complete revolution about the vertex of a ray.

Comparison of Degrees and Radians

There are 360° in one full circle and 2π radians in one full circle. It follows that

$$360^\circ = 2\pi \implies 1^\circ = \frac{\pi}{180}. \tag{1.1}$$

That is, 1° is equivalent to $\frac{\pi}{180}$ radians. But then we get the following relation:

$$n^\circ = n \cdot 1^\circ = n\frac{\pi}{180}, \tag{1.2}$$

i.e. n degrees is equal to $n\frac{\pi}{180}$ radians. But then we can divide both sides of equation (2) by $\frac{\pi}{180}$ to obtain

$$n\frac{180^\circ}{\pi} = n. \tag{1.3}$$

That is, n radians is equal to $n \cdot \frac{180}{\pi}$ degrees. Thus we have the following conversion:

Conversion from Degrees to Radians and Radians to Degrees

(Degrees $\longrightarrow$ Radians) To convert n° to radians, multiply n by $\frac{\pi}{180}$.

(Radians $\longrightarrow$ Degrees) To convert n radians to degrees, multiply n by $\frac{180}{\pi}$.

Example 2

Convert $\theta = -60^\circ$ to radians.

Solution Using the above conversion we multiply -60 by $\frac{\pi}{180}$ to get the corresponding radian measure. In particular, $\theta = -\frac{\pi}{3}$.

Example 3

Convert $\theta = \frac{4\pi}{3}$ to degrees.

Solution Using the above conversion we multiply $\frac{4\pi}{3}$ by $\frac{180}{\pi}$ to get 240°. Alternatively, one could have noticed that by example 2 we already knew that $\frac{\pi}{3}$ is 60° and hence 4 times $\frac{\pi}{3}$ must be 240°.

Recall that the Cartesian plane is consists of four quadrants. The angle θ lies in one of these quadrants or on one of the axes. We have the following:

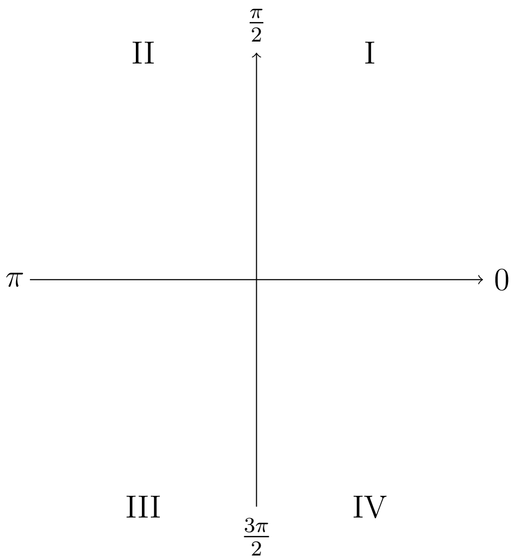
1) $0 < \theta < \pi/2 \implies \theta$ terminates in quadrant I,

2) $\pi/2 < \theta < \pi \implies \theta$ terminates in quadrant II,

3) $\pi < \theta < 3\pi/2 \implies \theta$ terminates in quadrant III,

4) $3\pi/2 < \theta < 2\pi \implies \theta$ teriminates in quadrant IV.

In particular, 90° corresponds to $\pi/2$ radians, 180° corresponds to π radians, and 270° corresponds to $3\pi/2$ radians.



Here is a useful formula for finding the area of a sector of a circle. Applying it is straightforward.

Proposition 1.1. Area of a Sector of a Circle *For a circle of radius r, the area A of a sector of the circle with central angle θ is given by*

$$A = \frac{1}{2}r^2\theta \tag{1.4}$$

where θ is measured in radians.

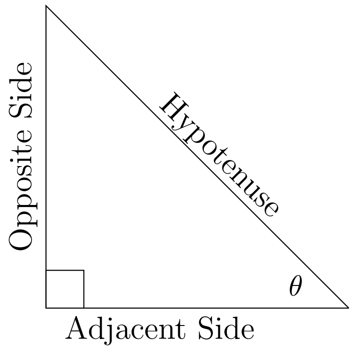
Proof. The total area of a circle is πr^2. Because the area of the sector is proportional to θ, and 2π is the angle of the whole circle, the area of the sector can be obtained by multiplying the circle's area by the ratio of the angle and 2π (i.e. the ratio of the"part" to the"whole"), that is $A = \pi r^2 \cdot \frac{\theta}{2\pi} = \frac{1}{2}r^2\theta$. □

We remark that we previously defined coterminal angles to be angles that have the same initial and terminal side. Observe that if you add any multiple of 360° to any angle θ that this gives a coterminal angle. This is how you should find coterminal angles.

1.2 Definitions of the Trigonometric Functions for Right Triangles

1.2.1 The Definitions

In this section we define the six trigonometric functions using right triangles. A *right triangle* is a triangle in which one of the angles measure 90° (or $\pi/2$ radians). An angle of measure 90° is called a *right angle*, an angle θ of measure $0° < \theta < 90°$ is called an *acute angle*, and an angle θ of measure $90° < \theta < 180°$ is called an *obtuse angle*. Relative to an angle θ in any triangle, the three sides of the triangle are the side *opposite* θ, the side *adjacent* to θ, and the *hypotenuse* of the triangle. Note that the hypotenuse of the triangle is always the longest side of the triangle and does not change relative to θ.



Definition 1.6. Right Triangle Definitions of the Trigonometric Functions Let θ be an *acute* angle of a right triangle. The six trigonometric functions of the angle θ are defined as follows:

$$\sin\theta = \frac{opp}{hyp} \qquad \cos\theta = \frac{adj}{hyp} \qquad \tan\theta = \frac{opp}{adj}$$

$$\csc\theta = \frac{hyp}{opp} \qquad \sec\theta = \frac{hyp}{adj} \qquad \cot\theta = \frac{adj}{opp}$$

where the abbreviations opp, adj, and hyp represent the *lengths* of the three sides of the triangles.

Remark. It is important to note that these trigonometric *functions* are *functions* in the same way that $f(x) = x^2$ is a *function*. They take a real number as an input and output another real number. Moreover, their graphs will pass the so called *vertical line test*.

From the above definition we obtain the following *trigonometric identities*.

Reciprocal Identities

$\sin u = \frac{1}{\csc u}$ $\quad \csc u = \frac{1}{\sin u}$,

$\cos u = \frac{1}{\sec u}$ $\quad \sec u = \frac{1}{\cos u}$,

$\tan u = \frac{1}{\cot u}$ $\quad \cot u = \frac{1}{\tan u}$.

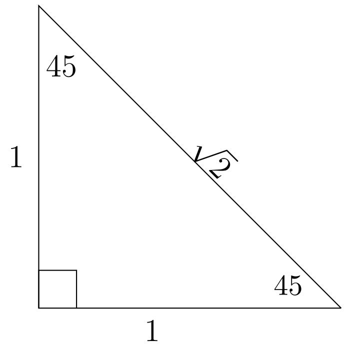
Quotient Identities

$\tan u = \frac{\sin u}{\cos u}$,

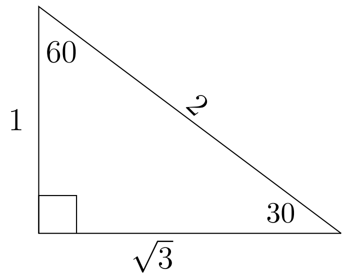
$\cot u = \frac{\cos u}{\sin u}$.

1.2.2 Special Right Triangles

There are two *special right triangles*, namely the $30 : 60 : 90$ and $45 : 45 : 90$, i.e. they have angles of measure $30°, 60°$ and $90°$ and similarly for the $45 : 45 : 90$. These triangles are invaluable for computing the value of the trigonometric functions for any angle, as we will see later.



Similarly, the 30 : 60 : 90 looks like:



1.3 Trigonometric Functions of Any Angle

1.3.1 Introduction

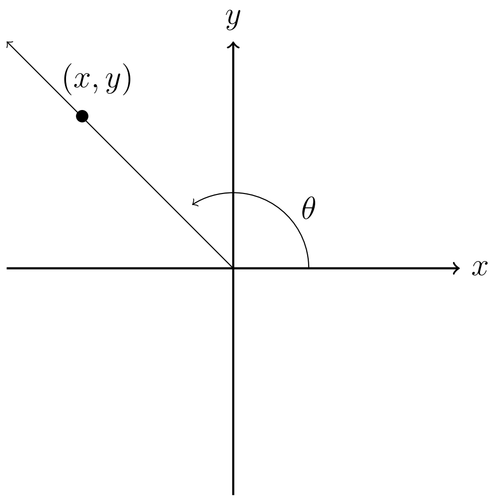
In the previous section we defined the trigonometric functions in terms of right triangles. This required the angle to be part of some right triangle, and right triangles only have a right angle and acute angles. This seems to pose a problem, namely *how can we evaluate the trigonometric functions for angles that are not acute?* So in this section we give an alternate definition for the trigonometric function which will work for *any angle*.

Definition 1.7. Definitions of Trigonometric Functions of Any Angle Let θ be an angle in standard position with (x, y) a point on the terminal side of θ and $r = \sqrt{x^2 + y^2} \neq 0$ (i.e. r is the distance the point (x, y) is from the origin).

Then the trig functions are defined as follows:

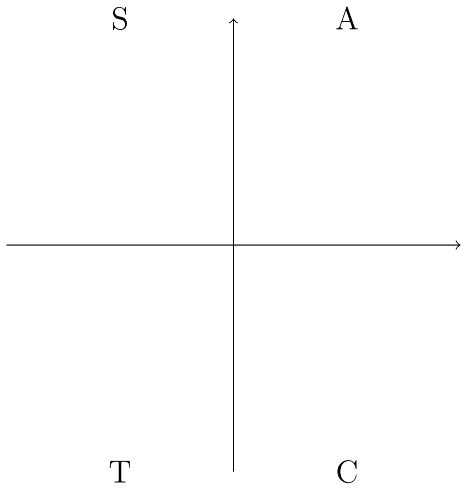
$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}, \quad x \neq 0$$

$$\csc\theta = \frac{r}{y}, \quad y \neq 0 \qquad \sec\theta = \frac{r}{x}, \quad x \neq 0 \qquad \cot\theta = \frac{x}{y}, \quad y \neq 0.$$

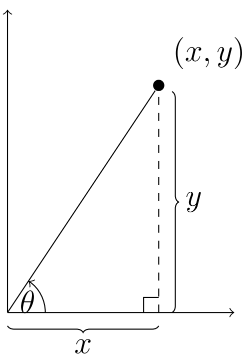


Using these definitions of the trigonometric functions, one easily sees which quadrants the trigonometric functions are positive and negative. For example, x and y are positive in quadrant I so *all* of the trigonometric functions are positive in quadrant I. However, in quadrant II x is negative and y is positive; it follows cosine and secant are the only two that are positive when the angle terminates in quadrant II.

The acronym ASTC corresponding to the picture below is often useful for remembering this. In the acronym the "A" means in quadrant I all of the trig functions are positive. The "S" means sine and its reciprocal, cosecant, are the only positive trig functions in quadrant II. Similarly, T means tangent and its reciprocal function, cotangent, are the only positive trig functions in quadrant III. Finally, the C means cosine and its reciprocal function, secant, are the only positive trig functions in quadrant IV.



We observe that this new definition of the trigonometric functions agrees with the definitions for the the trigonometric functions on right triangles if $0° < \theta < 90°$. To see this, suppose we have a point (x, y) lying on the terminal side of such an angle. In particular, we have the situation pictured below.



In this case x is the length of the adjacent side of the right triangle formed by the angle θ and the x-axis and, similarly, y is the length of the side opposite θ and r is the length of the hypotenuse.

1.3.2 Reference Angles

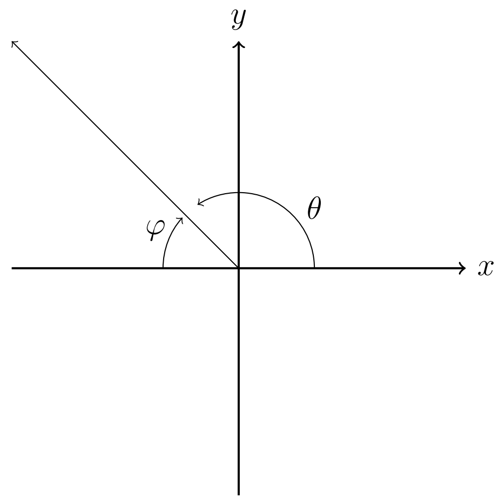
When θ does not terminate in quadrant I, we can still use the right triangle definitions of trigonometric functions to find their values using *reference angles*.

Definition 1.8. The *reference angle* of an angle θ in standard position is the acute angle φ formed by the terminal side of θ and the horizontal axis.

Remark. Reference angles are always positive angles.

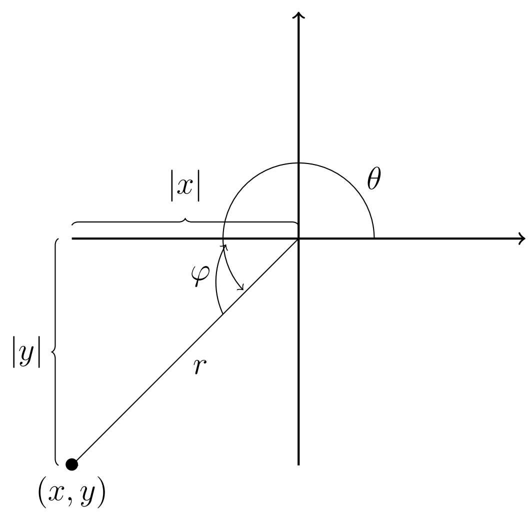
Example 1

In the pictures below, φ is the reference angle for the angle θ.



Example 2

This next example illustrates how we will, in general, use reference angles to evaluate a trigonometric function. Suppose we have an angle θ with reference angle φ as shown below.



Here the right triangle formed by the reference angle φ and the negative x-axis has (with respect to φ) opposite length $|y|$ and adjacent length $|x|$. Hence we have

$$\sin\varphi = \frac{|y|}{r}, \qquad \cos\varphi = \frac{|x|}{r}, \qquad \tan\varphi = \frac{|y|}{|x|}.$$

Using definition 0.7, we see that

$$\sin\theta = \frac{y}{r}, \qquad \cos\theta = \frac{x}{r}, \qquad \tan\theta = \frac{y}{x}.$$

Hence using the reference angle φ to find the value of the trigonometric function for the angle θ gives the correct answer up to sign; for example, we may have $\sin\varphi = -\sin\theta$. This is easily remedied by just remembering which quadrant θ terminates in and computing the trigonometric functions using the reference angle, and then making it positive or negative according to which quadrant θ terminates in. For example, if θ terminates in quadrant III, computing $\cos\varphi$ will yield a positive answer, so we simply need to remember that cosine is negative in quadrant III and so the correct answer will be $\cos\theta = -\cos\varphi$.

Example 3

We can apply the method from Example 2 to find the value of $\sin 300$. Observe that $\theta = 300$ lies in quadrant IV and has reference angle $\varphi = 60$. We can calculate $\sin 60$ easily using the $30 : 60 : 90$ special right triangle. We see from this triangle that $\sin 60 = \frac{\sqrt{3}}{2}$. However, we must now remember that θ terminates in quadrant IV and the value of sine must be negative. Hence just tack on the negative sign, i.e. $\sin 300 = -\sin 60 = -\frac{\sqrt{3}}{2}$.

1.3.3 Unit Circle

Recall definition 1.7 of the trigonometric functions. We had the point (x, y) lie on the terminal side of θ and $r^2 = x^2 + y^2$. In particular, the point (x, y) lies on the circle of radius r centered at the origin (because the equation of such circle is $x^2 + y^2 = r^2$). As a special case of this, when $r = 1$ we have that the point (x, y) lies on what is called the *unit circle* given by the equation $x^2 + y^2 = 1$. In this special case, definition 1.7 becomes

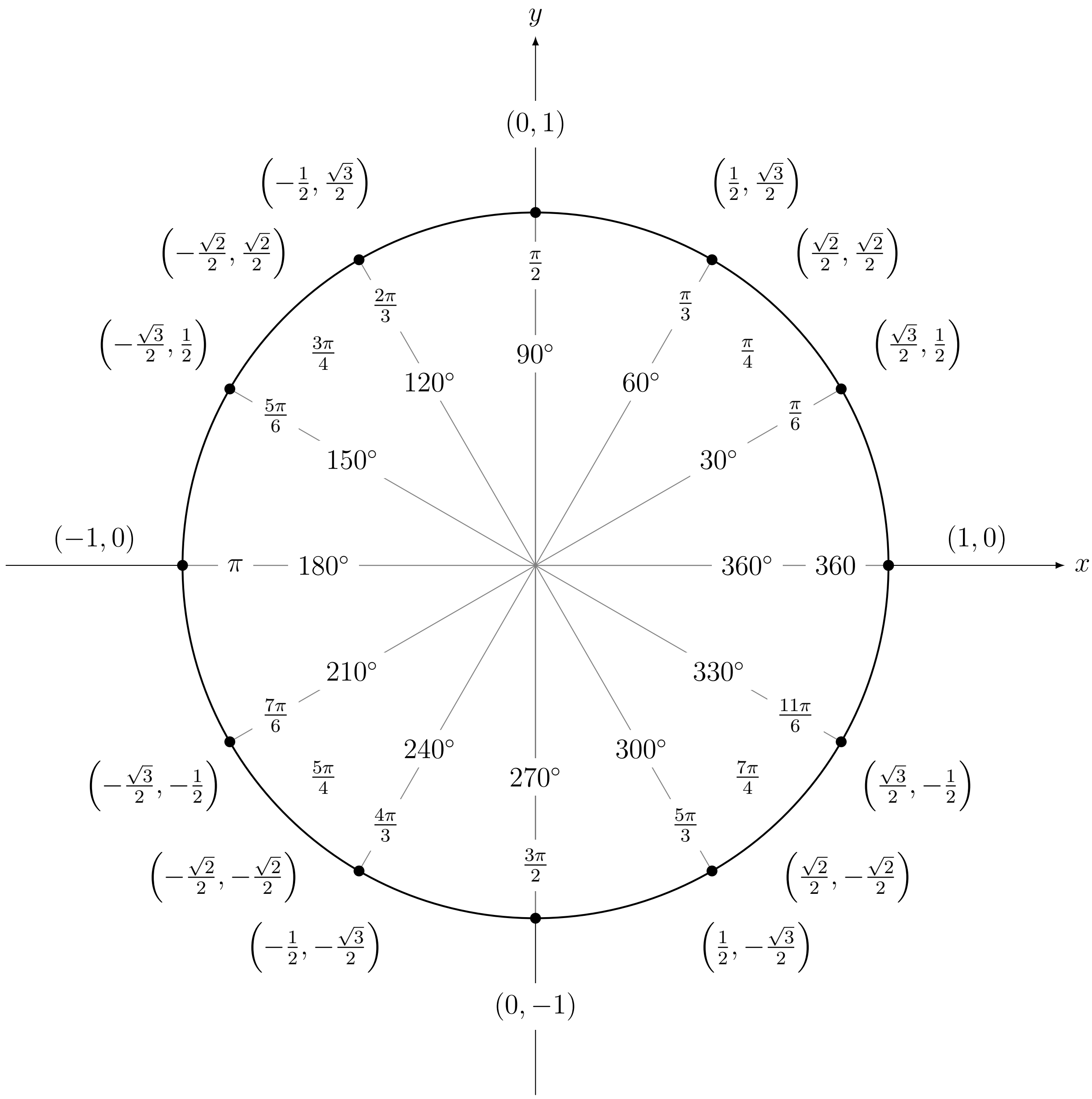
Definition 1.9. Definition of Trigonometric Functions on the Unit Circle

$$\sin\theta = y \qquad \cos\theta = x \qquad \tan\theta = \frac{y}{x}, \quad x \neq 0$$

$$\csc\theta = \frac{1}{y}, \quad y \neq 0 \qquad \sec\theta = \frac{1}{x}, \quad x \neq 0 \qquad \cot\theta = \frac{x}{y}, \quad y \neq 0.$$

The unit circle is nice because it gives us a compact way to remember some "special values" of the trigonometric functions (i.e. angles corresponding to special angles). These special angles are just the angles that correspond to an angle that terminates on one of the axis or an angle that has a reference angle of $30°, 45°$, or $60°$ which we can also easily compute the values of these angles using our special right triangles, as we saw in Example 3 above.

Since every angle on the unit circle either lies on one of the axes or has a reference angle of $30°, 45°$, or $60°$, we can also use the special right triangles from section 1.2.1 to find any value on the unit circle that does not lie on one of the axes (the points on the axes are easy to remember anyway). For example, $225°$ on the unit circle has reference angle $45°$. According to the circle, $\cos 225° - \frac{\sqrt{2}}{2}$. Using the $45 : 45 : 90$ special right triangle we have $\cos 45 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. Then we just need to remember that cosine is negative in quadrant III to conclude from the special right triangle that $\cos 225° = -\frac{\sqrt{2}}{2}$. As another example, $300°$ has reference angle $60°$ and, according to the unit circle, $\sin 300° = -\frac{\sqrt{3}}{2}$. Using the $30 : 60 : 90$ special right triangle to compute $\sin 60°$ gives $\sin 60° = \frac{\sqrt{3}}{2}$. Remembering that sine is negative in quadrant IV we conclude that $\sin 300° = -\frac{\sqrt{3}}{2}$.



1.3.4 Domain and Range of Sine, Cosine, and Tangent

Since $x^2 + y^2 = 1$, it follows that $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$; that is, $-1 \leq \sin\theta \leq 1$ and $-1 \leq \cos\theta \leq 1$. The domain is easily seen from definition 0.9 to be all real numbers $\mathbb{R}$. Tangent is not defined where $x = 0$. Hence it is easily seen by the unit circle that $\tan\theta$ is not defined for any $\theta = \frac{\pi}{2}k\pi$ where $n \in \mathbb{Z}$ (that is, if θ is any integer multiple of π). We also observe that as $x \to 0$; that is, as x gets close to zero, that tangent gets larger and larger (and in fact, gets as large as you want). To summarize, we have

Sine and Cosine: Domain = $\mathbb{R}$, Range = $[-1, 1]$,

Tangent: Domain = $\{x : x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\}$, Range = $\mathbb{R}$.

1.3.5 Some Identities

On the unit circle we have $x^2 + y^2 = 1$, the **Pythagorean Identities** follow immediately, that is,

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \cot^2\theta = \csc^2\theta \qquad 1 + \tan^2\theta = \sec^2\theta$$

Remark. The first identity follows immediately from definition 1.9, divide the first equation by $\sin^2\theta$ to obtain the second, and divide the first equation by $\cos^2\theta$ to obtain the last one.

One also sees the **even and odd** identities easily from the unit circle. They are

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta.$$

Remark. These are called even and odd identities because a function $f(x)$ is *even* if $f(-x) = f(x)$ and a function is *odd* if $f(-x) = -f(x)$. Hence sine, cosecant, tangent, and cotangent are *odd* whilst cosine and secant are *even.*

Definition 1.10. Periodic Function A function $f(t)$ is called a *periodic function* if there exists a positive real number k such that $f(t+k) = f(t)$ for all t in the domain of $f(t)$. The smallest number k for which $f(t)$ is periodic is called the *period* of $f(t)$.

Recall that there are 2π radians in a complete circle. Hence if have an angle θ which corresponds to a point (x, y) on the unit circle, then the point (x, y) also corresponds to the angle $\theta + 2\pi$ because we simply make a full circle back to it. It turns out that 2π is in fact the period of sine and cosine. However, tangent has period π because it repeats every half circle (i.e. every π radians). Hence we obtain the **periodic identities**

$$\sin(\theta + 2\pi) = \sin\theta \qquad \cos(\theta + 2\pi) = \cos\theta \qquad \tan(\theta + \pi) = \tan\theta$$

$$\csc(\theta + 2\pi) = \csc\theta \qquad \sec(\theta + 2\pi) = \sec\theta \qquad \cot(\theta + \pi) = \cot\theta.$$

The periodic identities of the trig functions are *very* useful. They assure us that given any angle θ, we can always reduce the problem to finding the value of the trig functions for an equivalent angle φ where $0 \le \varphi < 2\pi$. In particular, whenever you have an angle θ that is larger than 2π radians (or, equivalently, $360°$), you can compute the value of the trig function for the angle φ where φ is the remainder angle when you divide θ by 2π (or $360°$).

Example

Find the value of $\sin\frac{30721\pi}{2}$.

Solution: Observe that $30721\pi = 7680 \cdot 2\pi + \frac{\pi}{2}$. So we have $\sin\frac{30721\pi}{2} = \sin(7680 \cdot 2\pi + \frac{\pi}{2}) = \sin\frac{\pi}{2} = 1$ by the periodicity of sine.

1.4 Graphing Trigonometric Functions

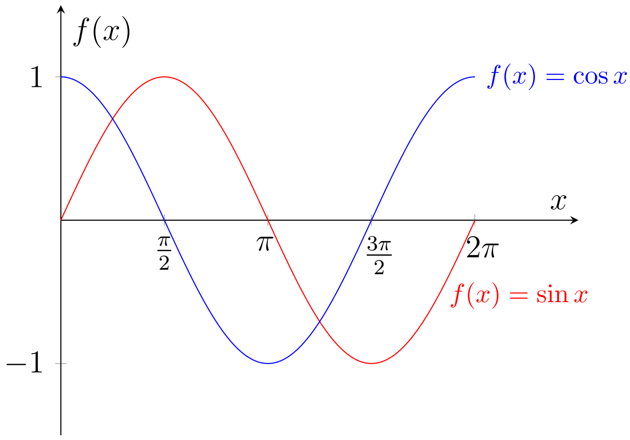
1.4.1 Introduction

In this section we will graph the main trigonometric functions, i.e. sine, cosine and tangent. Recall that the trigonometric functions are *periodic*, i.e. they repeat on intervals of certain length.

1.4.2 Graphing Sine and Cosine

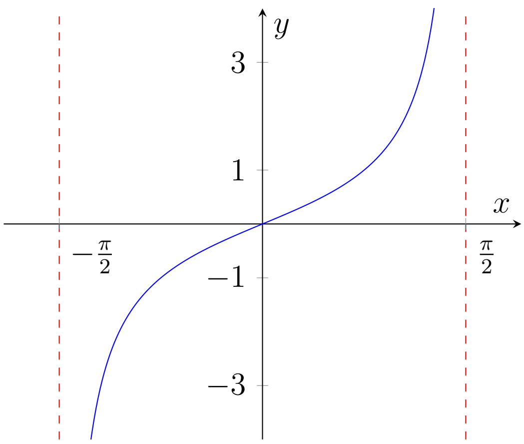
We will graph sine and cosine in their most simple form; that is, we graph $f(x) = \sin x$ and $f(x) = \cos x$. Graphing more general forms of sine and cosine will be done as examples in class. The period of sine and cosine is 2π so the standard graph will repeat on intervals of length 2π.

We observe that $\sin 0 = 0, \sin\frac{\pi}{2} = 1, \sin\pi = 0$, and $\sin\frac{3\pi}{2} = -1$. Similarly, $\cos 0 = 1, \cos\frac{\pi}{2} = 0, \cos\pi = -1$, and $\cos\frac{3\pi}{2} = 0$. These are the so called *key points* in this case. Having these values marked helps us to draw the graph. Sine and cosine are graphed below to illustrate how they should be connected.



1.4.3 Graphing Tangent

For tangent, we have $\tan(-\frac{\pi}{4}) = -1, \tan 0 = 0$ and $\tan\frac{\pi}{4} = 1$. Moreover, we have that $\tan(-\frac{\pi}{2})$ and $\tan\frac{\pi}{2}$ are undefined (because all points on the y-axis have x-coordinate 0) resulting in vertical asymptotes at these points. These are the key points for tangent on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ which is the standard interval for one period of tangent. The graph of tangent is below.



1.4.4 Transformations of the Sine and Cosine Graphs

Previously we graphed the most simple forms of sine and cosine, but we now consider functions of the form $f(x) = a\sin(bx - c) + d$ and $f(x) = a\cos(bx - c)$. In particular, we will see how the values $a, b, c,$ and d change the graphs of the functions from their simple forms. We will do all of the work for sine and the logic is literally the same for cosine. First consider the value a. Recall that

$$-1 \leq \sin x \leq 1.$$

Multiplying this inequality by a gives

$$-a \leq a\sin x \leq a$$

In particular, a changes the *range* of the sine (and cosine) function.

Definition 1.11. The quantity $|a|$ in $f(x) = a\sin x$ and $f(x) = a\cos$ is called the *amplitude*. It represents half the distance between the maximum and minimum values of the function.

Remark. If $a < 0$, i.e. is negative, then a not only changes the range, it also *reflects* the graph about the x-axis.

Now we consider the quantity b above. We know that $f(x) = a\sin x$ completes one period on the interval $[0, 2\pi]$, that is, for $0 \leq x \leq 2\pi$. It then follows that $a\sin bx$ completes one cycle for for $0 \leq bx \leq 2\pi$ that is, $f(x) = a\sin bx$ completes ones cycle on the interval

$$0 \leq x \leq \frac{2\pi}{b}.$$

Hence the period of $f(x) = a\sin bx$ is $\frac{2\pi}{b}$. Now consider what c does in the general equation $f(x) = a\sin(bx-c)$. We know from our transformations of functions that c is a *horizontal shift* of $s(x) = a\sin bx$ because $s(x-c) = f(x)$. As before, we know that $a\sin x$ completes one period for $0 \leq x \leq 2\pi$. Hence it follows that $a\sin(bx-c)$ completes one period on the interval $0 \leq bx - c \leq 2\pi$. Solving this inequality for x we see that $a\sin(bx-c)$ completes one period for

$$\frac{c}{b} \leq x \leq \frac{c}{b} + \frac{2\pi}{b}.$$

In particular, the period of $f(x) = a\sin(bx-c)$ is $\frac{2\pi}{b}$ and the graph is shifted (left or right) by $\frac{c}{b}$. The quantity $\frac{c}{b}$ is called the *phase shift*. To summarize, we have

Graphs of Sine and Cosine Functions
Assume that $b > 0$. The graphs of $f(x) = a\sin(bx-c)$ and $f(x) = a\cos(bx-c)$ have the following characteristics.

$$Amplitude = |a|, \qquad Period = \frac{2\pi}{b},$$

The left and right endpoints of a one period interval can be found by solving the inequality $0 \leq bx - c \leq 2\pi$.

1.5 Inverse Trigonometric Functions

Definition 1.12. A function is an *injective function* if $f(a) = f(b)$ implies $a = b$. In particular, for each point in the range of f there is only one point in the domain that maps to that point in the range, i.e. the f must pass the horizontal line test.

Definition 1.13. A function g is the inverse of the function f if $(f \circ g)(x) = (g \circ f)(x) = x$; that is, if $f(g(x)) = g(f(x))$. We usually denote the inverse of f by f^{-1}.

A function f has an inverse if and only if it is an *injective function*. The trigonometric functions are clearly not injective because they do not pass the horizontal line test.

Consider $y = \sin x$. If we restrict the domain of $\sin x$ to the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ then we see that $y = \sin x$ is injective on this domain and hence has an inverse here.

Definition 1.14. The *inverse sine function* is defined by $y = \sin^{-1} x$ if and only if $\sin y = x$ where $-1 \leq x \leq 1$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

The function $y = \sin^{-1} x$ is also often denoted $y = \arcsin x$ which is the exact same thing. The definition of the inverse sine function means that $y = \sin^{-1} x$ outputs the angle that you input into sine to get x on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Example 1

Given $f(x) = \sin^{-1} x$, find $f(\frac{\sqrt{3}}{2})$.

Solution: We have $f(\frac{\sqrt{3}}{2}) = \sin^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$ because $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Similar to how we restricted the domain of sine to an interval on which it is injective, we can restrict the domain of $\cos x$ to $0 \le x \le \pi$ and have cosine injective on this interval. Similarly, restrict $\tan x$ to the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ to get an injective function. The definitions of the the three inverse functions are compiled together below.

Definition 1.15. The Inverse Trigonometric Functions
The inverse trigonometric functions are defined as below. The domain of the functions is in the second column and the range of the functions is the third column.

$y = \sin^{-1} x \iff \sin y = x$	$-1 \le x \le 1$	$-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$
$y = \cos^{-1} x \iff \cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \tan^{-1} x \iff \tan y = x$	$-\infty < x < \infty$	$-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$

When you see the statement $\sin^{-1}(n)$, you should think of this as being "the angle whose sine value is equal to n".

Example 2

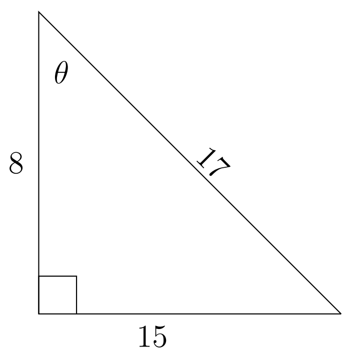
Find the value of $\tan(\cos^{-1}(\frac{1}{2}))$.

Solution: We recognize that $\cos 60 = \frac{1}{2}$ so that $\cos^{-1}(\frac{1}{2}) = 60$. Therefore $\tan(\cos^{-1}(\frac{1}{2})) = \tan(60) = \sqrt{3}$.

Example 3

Find the value of $\sin(\tan^{-1}(\frac{15}{8}))$.

Solution: Draw a triangle with angle θ so that $\tan\theta = \frac{15}{8}$ as below. We do this because then $\tan^{-1}(\frac{15}{8}) = \theta$. We recognize $8, 15, 17$ to be a Pythagorean triple. Alternatively we could have used the Pythagorean theorem to find the length of the hypotenuse. It follows then that $\sin(\tan^{-1}(\frac{15}{8})) = \sin\theta = \frac{15}{17}$.



The following inverse properties for functions hold for the trig functions in the correct domain and are often useful whilst solving certain equations.

Inverse Properties of Trigonometric Functions

1. If $-1 \leq x \leq 1$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, then

$\sin(\sin^{-1} x) = x$ and $\sin^{-1}(\sin y) = y$.

2. If $-1 \leq x \leq 1$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, then

$\cos(\cos^{-1} x) = x$ and $\cos^{-1}(\cos y) = y$.

3. If $x \in \mathbb{R}$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, then

$\tan(\tan^{-1} x) = x$ and $\tan^{-1}(\tan y) = y$.

Remark. These inverse properties do not hold for arbitrary values of x and y. For instance, $\sin^{-1}(\sin \frac{3\pi}{2}) = \sin^{-1}(-1) = -\frac{\pi}{2}$.

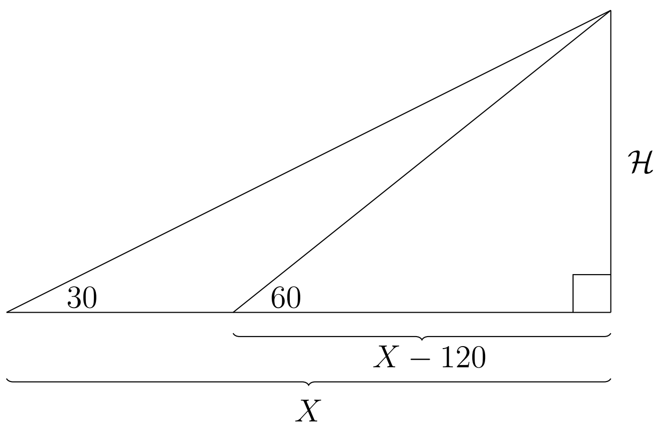
1.6 Applications of Trigonometric Functions

At this point we know enough trigonometry to do some application problems. This is best learned via examples so we will do several of them. To do these problems, it is important to be able to draw the correct picture.

Example 1

A surveyor measures the angle of elevation to the top of Mount Williams to be 30°. She then moves in a straight line to be 120 feet closer to the mountain. The angle of elevation is now 60°. How high is Mount Williams?

Solution: We can model this situation using a right triangle as shown below.



We call the initial distance from the surveyor to the base of the mountain distance X. Therefore, when she moves 120 feet closer to the mountain he is now $X - 120$ distance from the mountain. We wish to find the height $\mathcal{H}$. From the triangle we see that $\tan 30 = \frac{\mathcal{H}}{X}$ and $\tan 60 = \frac{\mathcal{H}}{X-120}$. Therefore we have

$$\mathcal{H} = X \tan 30 \tag{1.5}$$

and

$$\mathcal{H} = (X - 120) \tan 60.$$

Setting these two expressions for $\mathcal{H}$ equal yields

$$X \tan 30 = (X - 120) \tan 60.$$

Solving for X we have

$$X = \frac{-120 \tan 60}{\tan 30 - \tan 60} = 180.$$

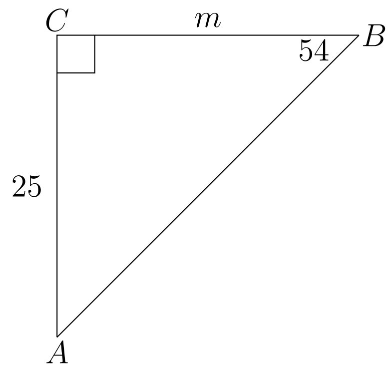
Substituting this value for X into equation 0.5 yields

$$\mathcal{H} = (180) \tan 30 \approx 104 \text{ feet.}$$

Example 2

A boat leaves port A and travels 25 miles straight north to port C and then turns east and travels m miles to port B. If the angle ABC measures 54°, then what is the distance m?

Solution: Again, we model this with a right triangle.



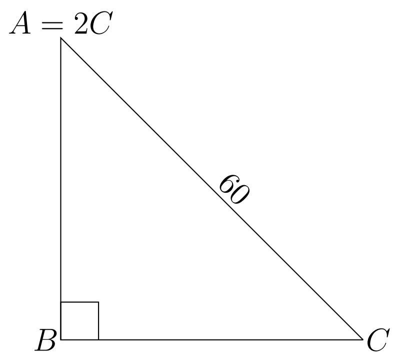
We have that

$$\tan 54 = \frac{25}{m} \implies m = \frac{25}{\tan 54} \approx 18.$$

Example 3

In a right triangle ABC, where angle B is the right angle, the measure of angle A is twice the measure of angle C. If side AC measures 60 picometers (pm), what is the length of side AB?

Solution: Model this again with a triangle as below.

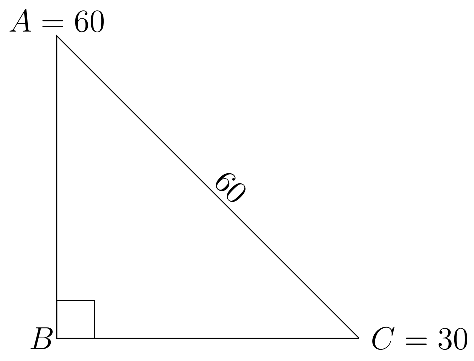


The critical observation here is that $A = 2C$. Furthermore, there are 180° in a triangle so we must have

$$2C + C + 90 = 180 \implies C = 30.$$

Since $C = 30$ it follows that $B = 60$.

We can relabel the triangle above.



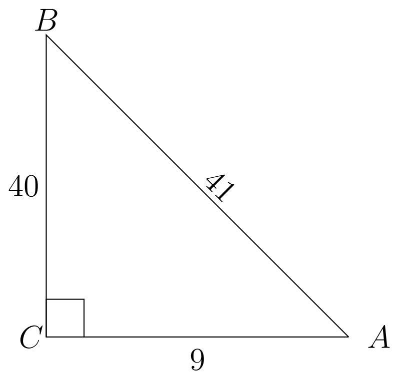
Let X denote the length of side AB. Then we have $\sin 30 = \frac{X}{60}$. Hence

$$X = 60 \sin 30 = 30 \text{ pm}.$$

Example 4

In right triangle ABC, with angle C being the right angle, $a = 40$ and $b = 9$. Find the length of side c and the measures of angles A and B.

Solution: Draw another triangle and label it appropriately. We immediately recognize $9, 40, 41$ as a Pythagorean triple so that we can label the hypotenuse as having length $c = 41$.



It is particularly simple to find the values of the angles because we know the lengths of the sides. We can use any of the trigonometric functions to do this, but we will use tangent. There is no particular reason to choose tangent over sine or cosine, but you have to choose one of them. We have

$$\tan B = \frac{9}{40} \implies B = \tan^{-1}(\frac{9}{40}) \approx 12.68^\circ.$$

Similarly,

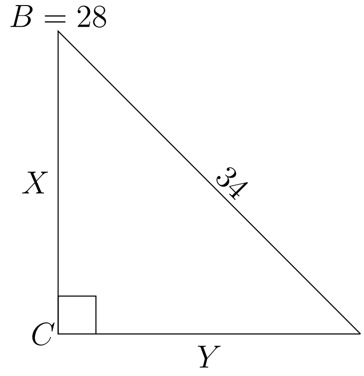
$$A = \tan^{-1}(\frac{40}{9}) \approx 77.31.$$

Alternatively, once we had angle B we could have just observed that $A = 180 - 90 - B$ because there are 180 degrees in a triangle.

Example 5

In right triangle ABC, with angle C being the right angle, side c measures 34 and angle B measures 28°. Find the perimeter of this triangle.

Solution: When the problem asks us to find the perimeter of the triangle, it is implicitly telling us to find the lengths of all three sides of the triangle. Draw the appropriate triangle as usual. We wish to find the values X and Y in our triangle below.



In contrast with the last problem, we only know the length of the hypotenuse which forces us to use sine or cosine to find the values of X and Y. We use cosine to find X and sine to find Y as follows:

$$\cos 28 = \frac{X}{34} \implies X = 34\cos 28 \approx 30.$$

Similarly, we have

$$\sin 28 = \frac{Y}{34} \implies Y = 34\sin 28 \approx 16.$$

1.6.1 Practice Problems

1) You are standing 45 meters from the base of the Empire State Building. You estimate that the angle of elevation to the top of the $86th$ floor is 82°. The total height of the building is another 123 meters above the $86th$ floor. What is the approximate height of the building? One of your friends is on the $86th$ floor. What is the distance between you and your friend?

2) You are skiing down a mountain with a vertical height of 1500 feet. The distance from the top of the mountain to the base is 3000 feet. What is the angle of elevation from the base to the top of the mountain?

3) A biologist wants to know the width w of a river in order to properly set instruments for studying the pollutants in the water. From point A, the biologist walks downstream 100 feet and sights to point C (i.e. he "looks at" point C). From this sighting, the biologist determines that $\theta = 54°$. How wide is the river?

4) In traveling across flat land, you notice a mountain directly in front of you. Its angle of elevation (to the peak) is 3.5°. After you drive 13 miles closer to the mountain, the angle of elevation is 9°. What is the height of the mountain?

5) *A Useful Approximation* The following approximation is often useful in physics. When θ is small, the following approximations can be used for sine and tangent:

$$\sin\theta \approx \theta, \tag{1.6}$$

$$\tan\theta \approx \theta, \tag{1.7}$$

where θ is in radians. The approximation for sine has less than 1% error when $\theta \le 0.176$ radians (i.e. $\theta \le 10°$), and the approximation for cosine has less than 1% error when $\theta \le 0.244$ radians (i.e. $\theta \le 14°$). These approximations are often useful in physics problems.

a) Show that these approximations hold.

b) Check for yourself that these work. Try computing, say, $\sin 0.10$ and $\tan 0.10$ on your calculator (make sure you have your calculator set to radians).

Hint: When θ is small, the arc made by the initial side of θ (i.e. the side adjacent to θ) is approximately equal to the length of the side opposite θ. In particular, if y is the side opposite θ, and s is the length of the arc intercepted by θ, then $s = x\theta \approx y$ where x is the length of the side adjacent to θ on the triangle.