2 Trigonometric Identities

We have already seen most of the *fundamental trigonometric identities*. There are several other useful identities that we will introduce in this section. We will see many applications of the trigonometric identities via examples in this section.

2.1 Fundamental Trigonometric Identities

Here are the the fundamental trigonometric identities compiled into one list.

Reciprocal Identities

$\sin u = \frac{1}{\csc u} \qquad \csc u = \frac{1}{\sin u}$,

$\cos u = \frac{1}{\sec u} \qquad \sec u = \frac{1}{\cos u}$,

$\tan u = \frac{1}{\cot u} \qquad \cot u = \frac{1}{\tan u}$.

Remark. Observe that the equations in each row are not actually different.

Quotient Identities

$\tan u = \frac{\sin u}{\cos u}$,

$\cot u = \frac{\cos u}{\sin u}$.

Pythagorean Identities

$\sin^2 u + \cos^2 u = 1$,

$1 + \tan^2 u = \sec^2 u$,

$1 + \cot^2 u = \csc^2 u$.

Cofunction Identities

$\sin(\frac{\pi}{2} - u) = \cos u \qquad \cos(\frac{\pi}{2} - u) = \sin u$,

$\sec(\frac{\pi}{2} - u) = \csc u \qquad \csc(\frac{\pi}{2} - u) = \sec u$,

$\tan(\frac{\pi}{2} - u) = \cot u \qquad \cot(\frac{\pi}{2} - u) = \tan u$.

Remark. These also work with degrees, i.e. by replacing $\frac{\pi}{2}$ with 90.

Even/Odd Identities

$\sin(-u) = -\sin u \qquad \csc(-u) = -\csc u$,

$\cos(-u) = \cos u \qquad \sec(-u) = \sec u$,

$\tan(-u) = -\tan u \qquad \cot(-u) = -\cot u$.

Remark. The only identities we have not seen previously are the *cofunction identities*. These are easily seen by drawing a right triangle. A the sum of all angles in a triangle is 180°. Hence in a triangle you have the right angle R, an acute angle θ, and that means that the third angle must be $90-\theta$ (or $\frac{\pi}{2}-\theta$ in radians). From this you should be able to convince yourself that the cofunction identities are true.

2.2 Trigonometric Equations

We can also use the fundamental identities to solve trig equations. Here are some examples. Note that the examples are all done in radians but we can work in degrees also when we do these problems.

Example 1

Solve $\cos x = \frac{\sqrt{2}}{2}$ for $x \in [0, 2\pi)$.

Solution: We know that $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ so $x = \frac{\pi}{4}$ is a solution. However, an angle with reference angle $\frac{\pi}{4}$ and terminating in a quadrant where cosine is positive (quadrant IV) will also have a value of $\frac{\sqrt{2}}{2}$. This is $x = \frac{7\pi}{4}$ which is the other solution. Thus the solution set is $x \in \{\frac{\pi}{4}, \frac{7\pi}{4}\}$.

Example 2

Solve $4\sin^2 x = 1$ for $x \in [0, 2\pi)$.

Solution: We have

$$4\sin^2 x = 1 \implies \sin^2 x = \frac{1}{4} \implies \sin x = \pm\frac{1}{2}.$$

Now using the same logic as in example 1, we conclude that $\sin x = \frac{1}{2}$ at $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$. Also, $\sin x = -\frac{1}{2}$ at $x = \frac{7\pi}{6}$ and $x = \frac{11\pi}{6}$. Hence the solution set is $x \in \{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\}$.

Example 3

Solve $\cos^2 x - 7\cos x + 12 = 0$ for $x \in [0, 2\pi)$.

Solution: Observe that this is a quadratic equation in cosine. Hence we can factor it using the usual factoring procedure for quadratics. That is, find numbers whose product is 12 and sum is -7. We have

$$\cos^2 x - 7\cos x + 12 = (\cos x - 4)(\cos x - 3) = 0.$$

Therefore we get the equations $\cos x - 4 = 0$ or $\cos x - 3 = 0$. The first equation gives $\cos x = 4$ and the second equation gives $\cos x = 3$. Since $-1 \leq \cos x \leq 1$ there is no solution to these equations.

Example 4

Solve the equation $\tan 2x = 1$ for $x \in [0, 2\pi)$.

Solution: We will use the inverse tangent function to solve this equation as follows: we assume that $\tan 2x$ lies in the domain of the inverse tangent function so that $\tan^{-1}(\tan 2x) = 2x$. Therefore

$$\tan 2x = 1 \implies \tan^{-1}(\tan 2x) = \tan^{-1}(1) \implies 2x = \frac{\pi}{4}.$$

Using the inverse functions only give us one of the solutions to this equation, but the inverse function tells us in this case that the angle $2x$ must have reference angle $\frac{\pi}{4}$. Additionally, $\tan 2x$ must be positive in the quadrant that $2x$ terminates in. This is quadrant III, so we get the second solution is $2x = \frac{5\pi}{4}$. Solving these equations for x we get the solution set $x \in \{\frac{\pi}{8}, \frac{5\pi}{8}\}$.

Example 5

Solve the equation $\frac{1+\sin x}{\cos x} = 1$ for $x \in [0, 2\pi)$.

Solution: Squaring both sides of the equation gives

$$(\frac{1+\sin x}{\cos x})^2 = 1^2 \implies \frac{1+2\sin x+\sin^2 x}{\cos^2 x} = 1 \implies 1+2\sin x+\sin^2 x = \cos^2 x.$$

In the last equation we have sines and cosines both appear; this is bad. Generally we want to have only one trig function in the equation. Think about how we can write this equation with only sines or cosines using some trig identity. The identity that will help us here is the Pythagorean identity since we can rewrite the last equation as

$$(1-\cos^2 x) + 2\sin x + \sin^2 x = 0.$$

But $1-\cos^2 x = \sin^2 x$ by the Pythagorean identity. So we rewrite this as

$$\sin^2 x + 2\sin x + \sin^2 = 2\sin^2 x + 2\sin x = 0 \implies \sin^2 x + \sin x = 0.$$

Factoring the last equation we get

$$\sin x(\sin x + 1) = 0.$$

Hence $\sin x = 0$ or $\sin x = -1$. This gives the solution set $x \in \{0, \frac{3\pi}{2}\}$.

Finally, remember that whenever you square equations you can introduce extraneous solutions. Therefore, you should substitute the values of x that you found back into the original equation to see if they are actually solutions to the equation. If you get an answer like $1 = -1$, then that value is not a solution and should be disregarded.

2.3 Sum and Difference Formulas

In this section and the one to follow we will prove the *Sum and Difference Formulas* and the *Half-Angle formulas*. The Sum and Difference Formulas are not trivial to prove geometrically so we will use *Euler's formula*. Euler's formula is one of the most famous equations in mathematics and you may have seen it before. The proof of Euler's formula unfortunately requires calculus so we cannot prove it, so we are going to assume it is true and use it to prove the Sum and Difference Formulas. Euler's formula says

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{2.1}$$

Now, consider $e^{i(u+v)}$. By Euler's formula we have that

$$e^{i(u+v)} = \cos(u+v) + i\sin(u+v). \tag{2.2}$$

But also,

$$\begin{aligned} e^{i(u+v)} &= e^{iu+iv} = e^{iu}e^{iv} = (\cos u + i\sin u)(\cos v + i\sin v) \\ &= (\cos u\cos v - \sin u\sin v) + (\cos u\cos v + \sin u\sin v)i \end{aligned} \tag{2.3}$$

Recall that two complex numbers are equal if and only if their real and imaginary parts are equal. Since equation **??** is equal to equation **??** above, this means that $\cos(u+v) = \cos u\cos v - \sin u\sin v$ and $\sin(u+v) = \cos u\cos v + \sin u\sin v$.

Similarly, using the even and odd identities we have

$$e^{i(u-v)} = \cos(u-v) + i\sin(u-v). \tag{2.4}$$

But also,

$$e^{i(u-v)} = e^{iu-iv} = e^{iu}e^{i(-v)} = (\cos u + i\sin u)(\cos(-v) + i\sin(-v))$$
$$= (\cos u + i\sin u)(\cos v - i\sin(v))$$
$$= (\cos u\cos v + \sin u\sin v) + (\sin u\sin v - \cos u\sin v)i. \tag{2.5}$$

Hence comparing equation **??** to **??** we have $\cos(u-v) = \cos u\cos v + \sin u\sin v$ and $\sin(u-v) = \sin u\sin v - \cos u\sin v$.

Therefore we have derived the sum and difference formulas. In summary,

Sum and Difference Formulas

$\sin(u \pm v) = \sin u\cos v \pm \cos u\sin v$

$\cos(u \pm v) = \cos u\cos v \mp \sin u\sin v$

$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u\tan v}$

Remark. Since $\tan(u \pm v) = \frac{\sin(u\pm v)}{\cos(u\pm v)}$ you can obtain the formula for tangent by dividing identity the first identity by the last one.

We will mostly use these identities to find values of trigonometric functions of angles that are not on the unit circle (i.e. angles that are not "special angles". For example, the angle $15°$ is not on the unit circle, but $15° = 45° - 30°$ and both $45°$ and $30°$ are on the unit circle. Therefore, we can find the exact value of sine and cosine of $15°$ by using $\sin(45-30)$ and $\cos(45-30)$ and using the sum and difference formulas.

2.4 Double-Angle and Half-Angle Formulas

The double angle trig identities are for $\sin 2u, \cos 2u$, and $\tan 2u$. These identities follow easily from the sum and difference formulas as follows. Using the sum and difference formulas, we have

$$\sin 2u = \sin(u+u) = \sin u\cos u + \cos u\sin u = 2\sin u\cos u. \tag{2.6}$$

Similarly,

$$\cos 2u = \sin(u+u) = \cos u\cos u - \sin u\sin u = \cos^2 u - \sin^2 u. \tag{2.7}$$

Since $\cos^2 u = 1 - \sin^2 u$ by the Pythagorean identity, substituting this value in for $\cos^2 u$ in (7) we get $\cos 2u = \cos^2 u - \sin^2 u = \cos^2 1 - 2\sin^2 u$. Also, by the Pythagorean identity $\sin^2 u = 1 - \cos^2 u$ so we also have $\cos 2u = \cos^2 u - \sin^2 u = 2\cos^2 u - 1$. As usual $\tan 2u = \frac{\sin 2u}{\cos 2u}$.

Double-Angle Formulas

$\sin 2u = 2\sin u\cos u,$

$\cos 2u = \cos^2 u - \sin^2 u = 2\cos^2 u - 1 = 1 - 2\sin^2 u,$

$\tan 2u = \frac{2\tan u}{1-\tan^2 u}.$

From the above equations you can easily obtain the so called *power-reducing formulas* which will be particularly useful in calculus. The sine power reducing formulas is obtained by solving the second double-angle formula for $\sin^2 u$ and the cosine power-reducing formulas are obtained from the double angle formulas by solving the second double-angle formula for $\cos^2 u$, and $\tan^2 u = \frac{\sin^2 u}{\cos^2 u}$ gives the power reducing formula for tangent.

Power-Reducing Formulas

$\sin^2 u = \frac{1-\cos 2u}{2}$,

$\cos^2 u = \frac{1+\cos 2u}{2}$,

$\tan^2 u = \frac{1-\cos 2u}{1+\cos 2u}$.

The last trigonometric identities that we need for this course are the *half-angle formulas*. They are obtained by replacing the angle u in the power-reducing formulas by half of the angle u, that is, the angle $\frac{u}{2}$. The half angle formulas allow us to find the values of some additional angles that are not on the unit circle.

Half-Angle Formulas

$\sin \frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$,

$\cos \frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$,

$\tan \frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$.

Whether you use the "plus" or "minus" version of the half-angle formulas is determined by which quadrant the angle $\frac{u}{2}$ lies in. For example, if u lies in quadrant four, then we know that $270° < u < 360° \implies 135° < \frac{u}{2} < 180°$, hence the angle $\frac{u}{2}$ lies in qudrant two. Therefore we know to use the positive version of the sine half-angle formula and the negative version of the cosine half-angle formula.