3 Topics in Trigonometry

3.1 Law of Sines

Previously, all of the triangles we have been dealing with have been *right triangles*. We will now learn how to solve *oblique triangles*.

Definition 3.1. An *oblique triangle* is a triangle with no right angles.

Given a triangle ABC with sides a, b, and c (opposite the corresponding angle as usual), we can solve it if we know the measure of one side and any two other measures of the triangle. In particular, we can solve the triangle if we know

1) Two angles and a side: AAS or ASA,
2) Two sides and an angle opposite one of them: SSA,
3) Three sides: SSS,
4) Two sides and their included angle: SAS.

Remark. This simply says we must know one side and two angles or two angles and one side.

Remark. Note that it is possible that given some side lengths and angles, it may be *impossible* to form a triangle with those side lengths an angles. If this happens, you will get some sort of contradiction, such as an undefined sine or cosine value or the angles you find don't add up to 180°.

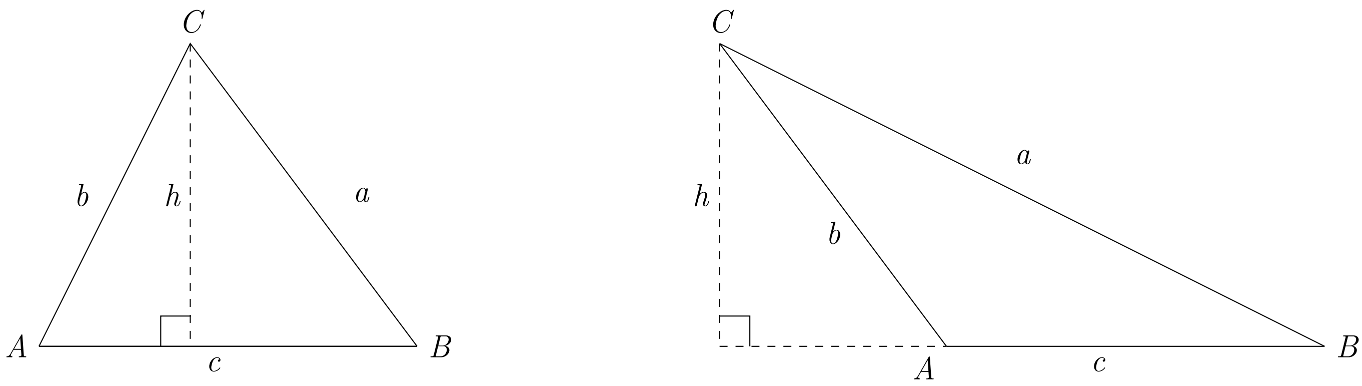
The *Law of Sines* will enable us to solve an oblique triangle in cases (1) and (2) whereas cases (3) and (4) will require the *Law of Cosines*.

Theorem 3.1. Law of Sines: *If ABC is a triangle (pictured below) with sides a, b, and c, then*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

The alternate form of the Law of Sines which is sometimes useful is the reciprocal form, namely

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$



Proof. Let h be the altitude of either triangle. Then you have

$$\sin A = \frac{h}{b} \implies h = b \sin A, \tag{3.1}$$

$$\sin B = \frac{h}{a} \implies h = a \sin B. \tag{3.2}$$

Hence setting these two equations equal yields

$$a \sin B = b \sin A \implies \frac{a}{\sin A} = \frac{b}{\sin B}. \tag{3.3}$$

Observe that $\sin A \neq 0$ and $\sin B \neq 0$ because no angle of a triangle has measure $0°$ or $180°$. You can similarly construct the altitude h from vertex B to side AC (extended in the obtuse triangle). Then you get

$$\sin A = \frac{h}{c} \implies h = c \sin A, \tag{3.4}$$

$$\sin C = \frac{h}{a} \implies h = a \sin C. \tag{3.5}$$

Equating those two values of h you obtain

$$a \sin C = c \sin A \implies \frac{a}{\sin A} = \frac{c}{\sin C}. \tag{3.6}$$

By transitivity of equality we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

□

The Ambiguous Case

If you are given two sides and an angle opposite one of them, i.e. you have the case SSA, then there are three possible situations.

a) No such triangle exists,

b) One such triangle exists,

c) Two distinct triangles may satisfy the conditions.

In case (a) you will get some sort of contradiction whilst working the problem, such as $\sin A$ is undefined. In case (c) there will be two angles between $0°$ and $180°$ which satisfy the conditions. In particular, in case (c), if $A = n°$ works as a value, then the the angle $180° - n°$ will also work because $\sin(180° - n°) = \sin(n°)$ when $0 < n < 90°$.

In the picture provided in the Law of Sines theorem, observe that the height of the triangle is $h = b \sin A$. We obtain from the usual area of a triangle formula that Area $= \frac{1}{2}(base)(height) = \frac{1}{2}(c)(b \sin A) = \frac{1}{2}bc \sin A$. Using the other altitudes of the triangle you can obtain the other two formulas in the following proposition.

Proposition 3.2. Area of an Oblique Triangle *The area A of any triangle is one-half the lengths of two sides times the sine of their included angle. That is,*

$$A = \frac{1}{2}bc \sin A = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B.$$

Hint. There are two correct answers.

3.2 Law of Cosines

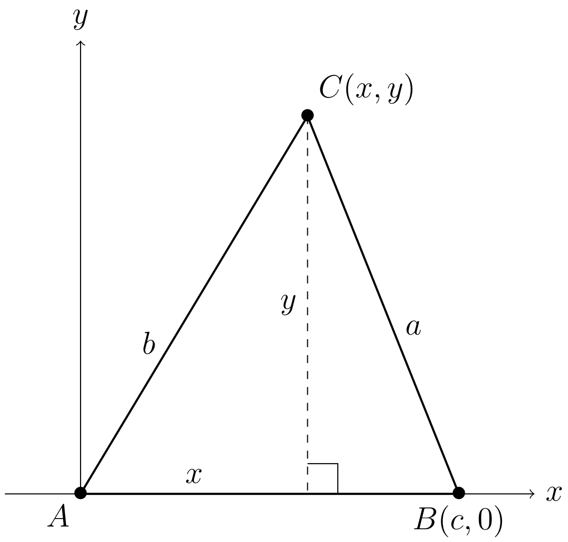
We saw in the previous section that we could use the Law of Sines to solve an oblique triangle in cases (1) and (2) above. The remaining cases are (3) and (4) which are SSS and SAS respectively. In these cases we must use the *Law of Cosines.*

Theorem 3.3. *Given oblique triangle ABC with sides $a, b,$ and c, the following relations hold*

$$a^2 = b^2 + c^2 - 2bc\cos A \implies \cos A = \frac{b^2+c^2-a^2}{2bc} \tag{3.7}$$

$$b^2 = a^2 + c^2 - 2ac\cos B \implies \cos B = \frac{a^2+c^2-b^2}{2ac} \tag{3.8}$$

$$c^2 = a^2 + b^2 - 2ab\cos C \implies \cos C = \frac{a^2+b^2-c^2}{2ab} \tag{3.9}$$



Remark. Observe that the Law of Cosines is a generalization of the Pythagorean Theorem. In particular, if the angle A,B, or C (depending on how you labeled the triangle) is 90° then the Law of Cosines reduces to the Pythagorean Theorem.

Proof. We will prove the first equation of the Law of Cosines. The other proofs are a matter of relabeling the triangle above. The triangle above has three acute angles. Vertex B has coordinates $(c, 0)$ and coordinate C has coordinates $(x, y) = (b\cos A, b\sin A)$. Moreover, it follows from the Pythagorean theorem that $a = \sqrt{(x-c)^2 + (y)^2}$. Hence we have

$$a^2 = (x-c)^2 + (y)^2$$

$$= (b\cos A - c)^2 + (b\sin A)^2$$

$$= b^2\cos^2 A - 2bc\cos A + c^2 + b^2\sin^2 A$$

$$= b^2(\sin^2 A + \cos^2 A) + c^2 - 2bc\cos A$$

$$= b^2 + c^2 - 2bc\cos A.$$

□

A corollary of the Law of Cosines is **Heron's Area Formula**. The proof of this is quite lengthy so we omit it here. The proof can be found in [**?**] or online.

Corollary 1. *Given any triangle with sides of lengths* a, b *and* c*, the area* A *of the triangle is*

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where

$$s = \frac{a+b+c}{2}.$$