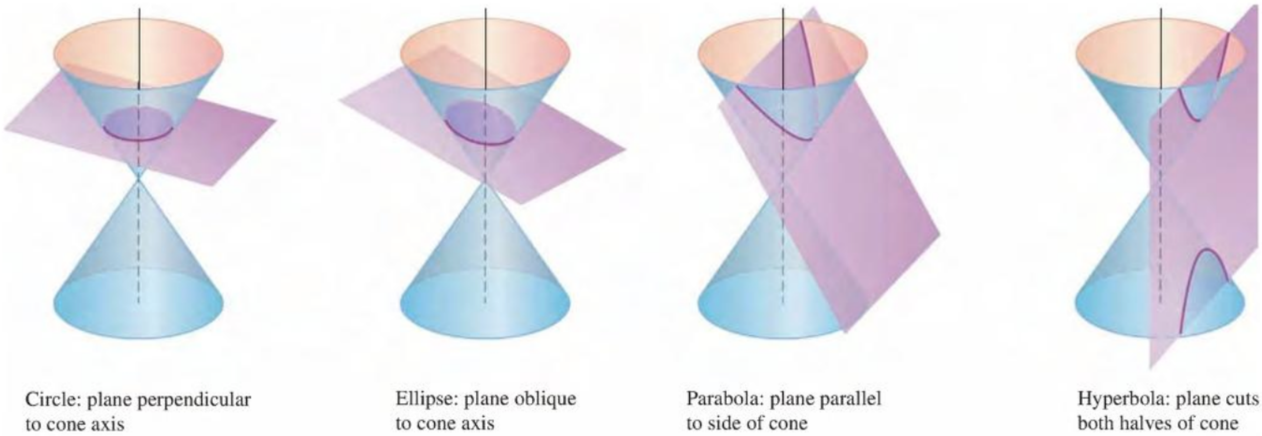
5 Introduction to Analytic Geometry: Conics

A *conic section* or *conic* is the cross section obtained by slicing a double napped cone with a plane not passing through the vertex. Depending on how you cut the plane through the cone, you will obtain one of three shapes, namely the *parabola*, *hyperbola*, or the *ellipse* and are show in Figure 1. These have interesting applications such as the discovery by Galileo that a projectile fired horizontally from the top of a tower falls to earth along a parabolic path. At around 1600 Kepler suggested that all planets move in elliptical orbits and, around 80 years later, this was shown to be true by Newton.

Figure 1: Planes Slicing a Double Napped Cone

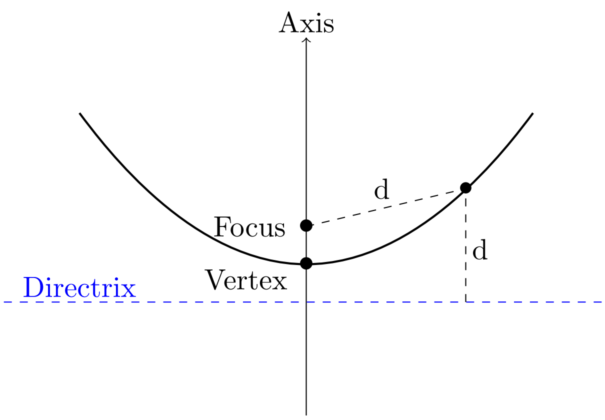


5.1 Parabolas

5.1.1 Definitions

In this section we will study *parabolas*. We already know that the graph of the quadratic function $f(x) = ax^2 + bx + c$ is a parabola that opens upward or downward. We can give a more general definition of a parabola that is independent of the orientation of the parabola (i.e. independent of whether the parabola opens upwards, downwards, sideways, etc.).

Definition 5.1. A *parabola* is the set of all points (x, y) in the plane that are equidistant from a fixed line, called the *directrix*, and a fixed point, called the *focus* that does not lie on the directrix. That is, if P is a point on the parabola, then the distance between P and the focus is equal to the distance between P and the directrix. The *vertex* is the midpoint between the focus and the directrix. The *axis* of the parabola is the line passing through the focus and the vertex.



Using the definition of a parabola, one can derive the *standard form of the equation of a parabola.*

The Standard Equation of a Parabola

The *standard form of the equation of a parabola* with vertex (h, k) and **vertical axis** (i.e. opens upward or downward) and **directrix** $y = k - p$ is given by

$$(x-h)^2 = 4p(y-k),\ \ p \neq 0, \tag{5.1}$$

and, similarly, for **horizontal axis** and **directrix** $x = h - p$ the equation is given by

$$(y-k)^2 = 4p(x-h),\ \ p \neq 0. \tag{5.2}$$

The focus lies on the axis p units from the vertex. In particular, in the case of the vertical axis, in which the axis passes through the point (h, k), the vertical axis is given by the equation $x = h$. Since the focus $\mathcal{F}_{\text{vert}}$ also lies on the line $x = h$ and is p units from the vertex, we must have that the focus is the point

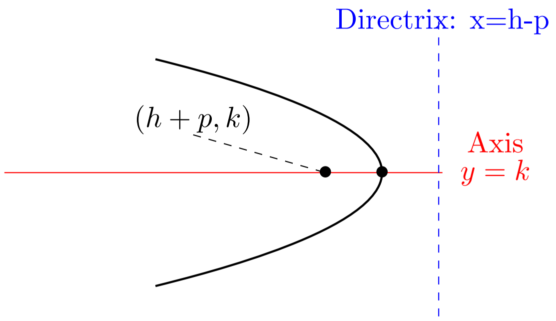
$$\mathcal{F}_{\text{vert}} = (h, k+p). \tag{5.3}$$

In the horizontal axis case, in which the axis passes through the point (h, k), the horizontal axis is given by the equation $y = k$. Since the focus $\mathcal{F}_{\text{horiz}}$ also lies on the line $y = k$ and is p units from the vertex, we must have that the focus is the point

$$\mathcal{F}_{\text{horiz}} = (h+p, k). \tag{5.4}$$

This situation is pictured after the following paragraph for the horizontal axis case with $p < 0$.

The parabola always opens away from the directrix and towards the focus similar to the image below. Hence as a general rule we have in the vertical axis case if $p > 0$ the parabola opens upward because the point $(h, k+p)$ lies above the vertex (h, k) and if $p < 0$ the parabola opens downward because the point $(h, k+p)$ will lie below the vertex. Similarly, in the horizontal case if $p > 0$ the parabola opens to the right because the point $(h+p, k)$ lies to the right of the vertex and if $p < 0$ the parabola opens to the left because the point $(h+p, k)$ lies to the left of the vertex.



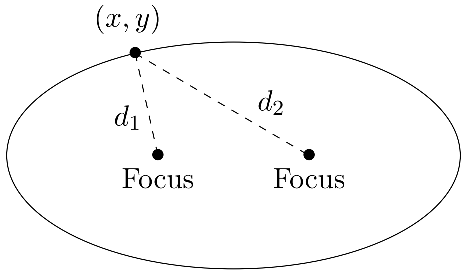
5.1.2 Derivation of the Standard Equation of a Parabola*

5.2 Ellipses

5.2.1 Definitions

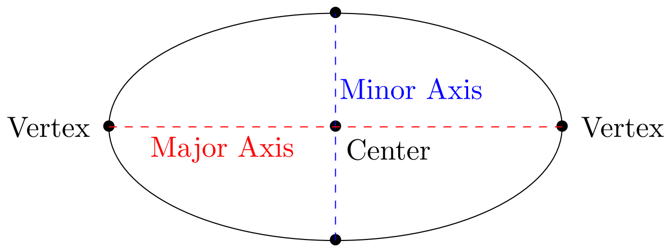
The second type of conic we study is the *ellipse*.

Definition 5.2. An *ellipse* is the set of all points (x, y) in the plane surrounding two foci such that the sum of the distances to each focus is constant for each point (x, y).



Remark. If (x, y) and (z, w) are two points on the ellipse and d_1 and d_2 is the distance from (x, y) to each each focus and d_1' and d_2' is the distance from (z, w) to each focus, then $d_1 + d_2 = d_1' + d_2'$ by definition.

Definition 5.3. The line through the two foci is called the *major axis* and the points on the ellipse that the major axis intersects are called the *vertices*. The line through the center that is perpendicular to the major axis is called the *minor axis*.



Using the definition of an ellipse one can derive the *standard equation of an ellipse*. The derivation is quite tedious so here we will just give the equation. If the reader is interested in the derivation it is in the next section but the reading is entirely optional.

The Standard Equation of an Ellipse

The *standard equation of an ellipse* with center (h, k) and major and minor axes of length $2a$ and $2b$, respectively, where $0 < b < a$, with **horizontal major axis** is given by

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

and, similarly, with **vertical major axis** is given by

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1.$$

The foci lie on the major axis, c units from the center, with $c^2 = a^2 - b^2$.

Remark. Note that if $a = b$ then the major axis and minor axis are equal length, which intuitively seems like a circle. Indeed, the above equations yield

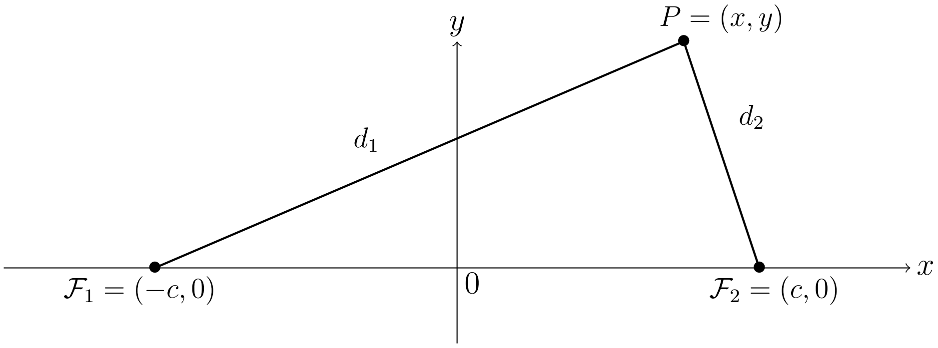
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2} = 1 \implies (x-h)^2 + (y-k)^2 = a^2$$

which the reader should recognize the equation of a circle centered at (h, k) with radius a.

5.2.2 Derivation of the Standard Equation of an Ellipse*

Here we will derive the standard equation of an ellipse with horizontal major axis using definition ??. The vertical major axis case is done analogously. Given the foci $\mathcal{F}_1$ and $\mathcal{F}_2$, we can introduce a coordinate system by placing the x-axis through the foci with the origin half way between. Let $\mathcal{F}_1 = (-c, 0)$ and $\mathcal{F}_2 = (c, 0)$ be the coordinates of the foci. If $P = (x, y)$ is an arbitary point in the plane, the distances d_1 and d_2 from $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, are given by the distance formula

$$d_1 = \sqrt{(x+c)^2 + y^2}$$
$$d_2 = \sqrt{(x-c)^2 + y^2}.$$



All points P lying on the ellipse are the constant distance $d_1 + d_2$ from $\mathcal{F}_1$ and $\mathcal{F}_2$. Denoting this constant by $2a$, we have

$$d_1 + d_2 = 2a$$

for all P on the ellipse. In order to ensure that the ellipse has points on it that do not lie on the line segment $\mathcal{F}_1\mathcal{F}_2$, we must have $d_1 + d_2 > 2c$, which is the distance between the foci. Note that $d_1 + d_2 > c$ implies that $a > c$. Now, substituting the values above for d_1 and d_2 into the equation $d_1 + d_2 = 2a$ we get

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a. \tag{5.5}$$

To simplify this equation, move the second radical to the right hand side, square the equation and simplify, isolate the remaining radical, and square the equation again. These manipulations will result in

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \tag{5.6}$$

Now we let $b = \sqrt{a^2 - c^2}$ and equation ?? becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is the equation of an ellipse in standard form centered at the origin.

Recall that $f(x-h)$ is a shift of the graph $f(x)$ by h units. Similarly, replacing x by $x - h$ and y by $y - k$ shifts this ellipse centered at the origin $(0, 0)$ to centered at (h, k) which is the general form of the ellipse.

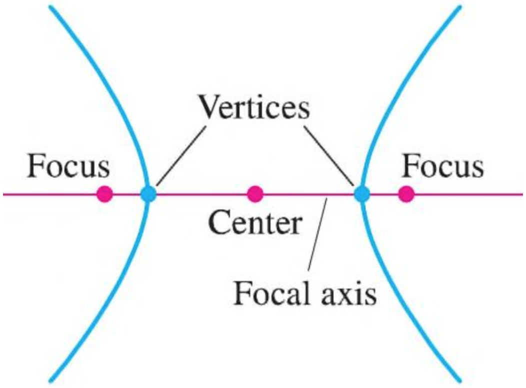
5.3 Hyperbolas

5.3.1 Definitions

The last type of conic is the *hyperbola* The hyperbola is similar to an ellipse in the sense that for an ellipse, the *sum* of the distances between the foci and a point on the ellipse is fixed, whereas for a hyperbola, the *difference* of the distances between the foci and a point on the hyperbola is fixed.

Definition 5.4. A *hyperbola* is the set of points (x, y) in a plane whose distances from two fixed points in the plane have a constant difference. The two fixed points are the *foci* of the hyperbola.

The line through the foci is called the *focal axis* or the *transverse axis* . The point on the axis halfway between the foci is the hyperbola's *center.* The points where the focal axis and the hyperbola cross are the *vertices.*



Standard Equation of a Hyperbola

The *standard form of the equation of a hyperbola* with center (h, k) with a **horizontal focal axis** is given by

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{5.7}$$

and with **vertical focal axis** is given by

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1. \tag{5.8}$$

The vertices are a units from the center, and the foci are c units from the center and $c^2 = a^2 + b^2$.

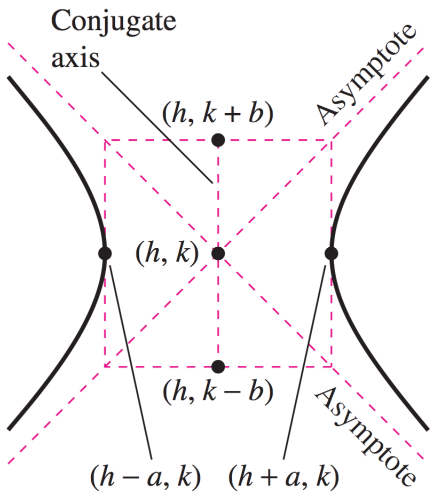
Example

Find the standard from of the equation of the hyperbola with vertices $(4, 1)$ and $(4, 0)$ and with foci $(4, 0)$ and $(4, 10)$.

solution: The vertices both have x-coordinate 4 and hence lie on the line $x = 4$. This means that the focal axis is vertical. The center is the midpoint between the vertices which, by the midpoint formula, we calculate the center to be $(4, 5)$. In particular, since the vertices have distance a units from the center we conclude that $a = 4$. Now, the foci are c units from the center, so we conclude that $c = 3$. We use the relationship $c^2 = a^2 + b^2 \implies b = \sqrt{c^2 - a^2}$ to determine that $b = 3$. Substituting these values into the equation for the hyperbola with vertical focal axis gives us

$$\frac{(y-5)^2}{16} - \frac{(x-4)^2}{9} = 1.$$

Remark. Drawing a picture is useful to help solve problems of this type.



The Asymptotes of a Hyperbola

Any hyperbola has two *asymptotes* that intersect at the center of the hyperbola. The asymptotes pass through the vertices of a rectangle centered at (h, k) with dimensions $2a$ and $2b$. The line segment of length $2b$ passing through the center of the hyperbola and joining the point $(h, k + b)$ and $(h, k - b)$ [or in the vertical axis case, joining $(h + b, k)$ and $(h - b, k)$] is called the *conjugate axis* of the hyperbola. Here are the equations for the asymptotes of a hyperbola.

Equations for Asymptotes of a Hyperbola

The asymptotes of a hyperbola with a **horizontal focal axis** are given by

$$y = k \pm \frac{b}{a}(x - h) \tag{5.9}$$

and for a **vertical focal axis** are given by

$$y = k \pm \frac{a}{b}(x - h). \tag{5.10}$$

Example

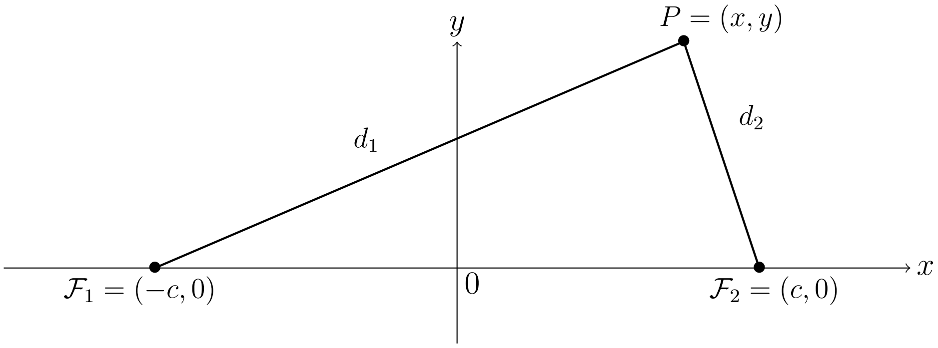
Put the equation $9x^2 - 16y^2 = 144$ of the hyperbola in standard form and find the hyperbola's asymptotes.

solution: Divide through by 144 to get $\frac{x^2}{16} - \frac{y^2}{9} = 1$. This equation is now in standard and we recognize it to be a hyperbola with horizontal focal axis centered at $(0, 0)$. Moreover, $a = 4$ and $b = 3$. Substituting these values into the equation for the horizontal focal axis we find that the asymptotes are

$$y = \pm\,\frac{3}{4}x.$$

5.3.2 Derivation of the Equation of a Hyperbola*

This derivation is very similar to the derivation of the equation for an ellipse and we do it for horizontal focal axis. The vertical focal axis case is done similarly. Suppose that the foci are $\mathcal{F}_1 = (-c, 0)$ and $\mathcal{F}_2 = (c, 0)$ as in the figure below.



The point $P = (x, y)$ lies on the hyperbola if the difference $d_1 - d_2$ remains constant, and we call this constant $2a$. Since $d_1 - d_2$ is possibly negative, this relation becomes $|d_1 - d_2| = 2a$ which is equivalent to $d_1 - d_2 = \pm 2a$. This is equivalent to the statement

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = \pm 2a$$

where we have used the distance formula to find d_1 and d_2 as we did in deriving the equation of the ellipse. Simplifying this equation similar to as in the ellipse case we obtain the equation

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1.$$

Now let $b = \sqrt{c^2 - a^2}$ and rewrite the previous equation as

$$\frac{x^2}{a^2} - \frac{y^2}{b^2}$$

which is the equation of a hyperbola with horizontal axis centered at $(0, 0)$. Making the transformation described at the end of the ellipse derivation yields the equation for a hyperbola centered at (h, k).

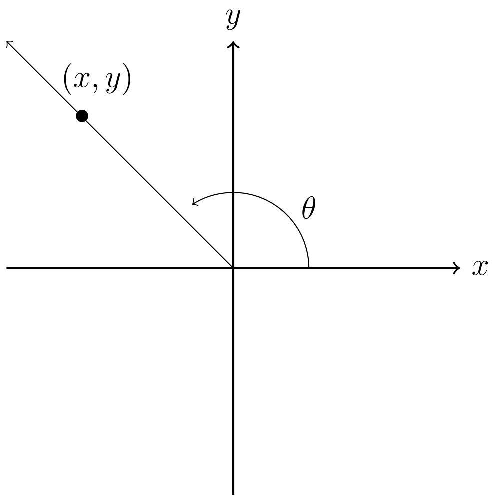
5.4 Polar Coordinates

5.4.1 Introduction

So far we have been representing graphs of equations in what is called the *Cartesian coordinate system* or *rectangular coordinate system*. In the Cartesian coordinate system we represent points as (x, y) where x and y represent the directed distances from the coordinate axes to the point (x, y). In particular, the point $(3, 4)$ means we go 3 units in the x direction and then 4 units in the y direction. This is not the only way that one can represent points in the plane. For example, if someone was standing at the origin of the plane, and was given a direction angle θ to the point (x, y), and also given the distance they would need to walk in order to get to the point (x, y), they should also be able to reach the point (x, y) in this way. This is the basic idea behind the polar coordinate system. In fact, this is essentially nothing that we have not studied before. In section 3.4 we represented complex numbers in polar form which was really just representing complex numbers in polar coordinates.

5.4.2 Representing Points in Polar Coordinates

To make mathematically precise what was said above, we wish to represent the point (x, y) as a point (r, θ) in the polar coordinate system. Suppose we have the point (x, y) in the plane and let θ be the angle from the origin to the ray passing through the point (x, y). Moreover, let $r = \pm\sqrt{x^2 + y^2}$ be the directed distance from the point (x, y) to the origin.



This is essentially the same setup as we had in the section 3.4, the only difference being that now we are in the Euclidean plane instead of the complex plane. But just as we did before, we write

$$x = r\cos\theta \tag{5.11}$$

$$y = r\sin\theta \tag{5.12}$$

and by dividing **??** by **??** we get the relationship

$$\tan\theta = \frac{y}{x}.$$

Remark. Negative values of r are also allowed since r gives a *directed distance*. If r is negative, the point $(r,\theta) = (r+\pi,\theta)$, or reflecting the point $(\|r\|,\theta)$ across the origin. It is usually best to find a polar representation of a point with $r > 0$.

Coordinate Conversion

The polar coordinates (r,θ) are related to the Cartesian coordinates (x,y) as follows:
To convert from **polar** → **Cartesian** we use the relationship

$$x = r\cos\theta$$

$$y = r\sin\theta$$

where $r^2 = x^2 + y^2$ and θ is the direction angle.

To convert from **Cartesian** → **polar** we use the relationship

$$\tan\theta = \frac{y}{x}$$

$$r^2 = x^2 + y^2.$$

Example 1: Cartesian → Polar

Represent the point $(\sqrt{3},-1)$ to polar coordinates.

solution: We wish to represent the point $(\sqrt{3},-1)$ in the form (r,θ) so we must find r and θ. We have $r = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = 2$. To find θ, note that the point $(\sqrt{3},-1)$ lies in quadrant IV and, moreover, $\tan 330° = \frac{\sqrt{3}}{3}$ so $\theta = 330°$. Hence we can represent $(\sqrt{3},-1)$ as $(2, 330°)$ in polar coordinates.

Remark. Note that the representation of points in polar coordinates is not unique. In particular, we used $\theta = 330°$ in the example above. However, any angle coterminal to $330°$ will work as well.

Example 2: Polar $\rightarrow$ Rectangular

Convert the point $(2, \pi)$ to Cartesian coordinates.

solution: This is very straightforward. We have $r = 2$ and $\theta = \pi$. Hence $x = 2\cos\pi$ and $y = 2\sin\pi$ so we have that the point in Cartesian coordinates is $(-2, 0)$.

Example 3

Convert the Cartesian equation $x^2 + y^2 = 9$ to polar form.

solution: Note that this is the equation of a circle with radius 3. To convert to polar coordinates we use the conversion $x = r\cos\theta$ and $y = r\sin\theta$ to get

$$(r\cos\theta)^2 + (r\sin\theta)^2 = r^2\cos^2\theta + r^2\sin^2\theta = 9$$

and factoring out the r^2 and using the fact that $\sin^2\theta + \cos^2\theta = 1$ we get the equation

$$r = 3$$

In particular, the polar equation of a circle has a very simple representation.

Example 4

Convert the polar equation $r = 4\sin\theta$ to Cartesian form.

solution These problems are pretty tricky. Multiply the equation $r = 4\sin\theta$ by r to get

$$r^2 = 4r\sin\theta.$$

Now use the fact that $y = r\sin\theta$ and $r^2 = x^2 + y^2$ to rewrite this as

$$x^2 + y^2 = 4y.$$

You can complete the square on this equation to write it as

$$x^2 + (y-2)^2 = 4$$

which we recognize as the equation of a circle of radius 2 centered at $(0, 2)$.