6 Sequences and Series

6.1 Introduction to Sequences and Series

6.1.1 Sequences

A good colloquialism for the definition of a sequence is that a sequence is an ordered list of number $a_1, a_2, ..., a_n, ...$ For example, $3, 6, 9, 12, ..., 3n, ...$ is a sequence with $a_1 = 3$, $a_2 = 6$,.., and $a_n = 3n$. The integer n is called the *index* of a_n. You can also use other letters like k or i or basically anything you want. We can think of a sequence as a function that has has its domain as the set of positive integers $\mathbb{Z}^+$ which maps n to a_n and this is the precise definition of a sequence.

Definition 6.1. An *infinite sequence* of numbers is a function whose domain is the set of positive integers $\mathbb{Z}^+$ or natural numbers $\mathbb{N}$. If a sequence has domain only the first n integers or natural numbers, then the sequence is called a *finite sequence.*

Sequences can be described by writing rules that specify their terms, such as $a_n = n^2$ or $b_n = (-1)^n \frac{1}{n}$.

Example 1: Writing Out the Terms of a Sequence

Write out the first five terms of the sequence $b_n = \frac{(-1)^n}{n}$.

Solution: Evaluate the expression for b_n for $n = 1$ to get $b_1 = \frac{(-1)^1}{1} = (-1)(1) = 1$. Similarly, we get

$$b_2 = \frac{1}{2},$$

$$b_3 = -\frac{1}{3},$$

$$b_4 = \frac{1}{4},$$

and

$$b_5 = -\frac{1}{5}.$$

The next example illustrates finding the general n^{th} term of a sequence given some of the terms of the sequence. This type of problem is pretty tricky but is useful and will come up in a differential equations course.

Example 2: Finding the n^{th} Term of a Sequence

a) For the sequence 1, -1, 1, -1,...

Solution: The sequence just alternates between $+1$ and -1. Since -1 raised to any even power gives positive 1 and -1 raised to any odd power gives -1 the sequence can be described as $a_n = (-1)^{n+1}$.

b) For the sequence $\frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \frac{8}{81}, ...$

Solution: Think about how the numbers are related to the index. We have $a_1 = \frac{1}{3}$, $a_2 = \frac{2}{9}$, $a_3 = \frac{4}{27}$ and $a_4 = \frac{8}{81}$. After awhile you should be able to see that the general formula is

$$a_n = \frac{2^{n-1}}{3^n}.$$

c) For the sequence $1+\frac{1}{2}$, $1+\frac{3}{4}$, $1+\frac{7}{8}$, $1+\frac{15}{16}$, $1+\frac{31}{32}$...

Solution: The 1 appears in every term, so it suffices to find the sequence a_n such that the sequence above is $b_n = 1 + a_n$. It is easy to see that the denominators of the fractions are described by 2^n. Moreover, the numerators are exactly 1 less than the denominators of the fractions, so you can write those as $2^n - 1$. Hence the sequence is $a_n = \frac{2^n-1}{2^n}$. In particular,

$$b_n = 1 + \frac{2^n - 1}{2^n}.$$

Example 3: The Fibonacci Sequence

You can also define a sequence *recursively* such as the Fibonacci sequence. This means that the general term for the sequence is defined based on previous terms of the sequence. The Fibonacci sequence is defined as follows:

$$a_0 = 1$$
$$a_1 = 1$$
$$\vdots$$
$$a_k = a_{k-2} + a_{k-1}.$$

To write down, say, the first five terms of the Fibonacci sequence, you would have to compute a_3, a_4, and a_5 using the a_k definition. We compute $a_3 = a_{3-2} + a_{3-1} = a_1 + a_2 = 1 + 1 = 2$. You can do the others similarly. If you like programming, it might be an amusing exercise to write a program that can compute any term of the Fibonacci sequence.

A special type of product called the *factorial* is important to know.

Definition 6.2. If n is a positive integer, we define n *factorial* as

$$n! = 1 \cdot 2 \cdot 3 \cdot ... \cdot (n-1) \cdot n.$$

We also define $0! = 1$.

The factorial $n!$ is simply the product of the first n integers, for example, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. Observe that $n! = n(n-1)!$. Hence we could have alternatively defined $n!$ recursively as follows:

$0! = 1,$
$n! = n(n-1)!$.

Example 4

a) Compute $\frac{5!}{8!}$.

Solution: Observe that $\frac{5!}{8!} = \frac{5!}{8 \cdot 7 \cdot 6 \cdot 5!} = \frac{1}{8 \cdot 7 \cdot 6} = \frac{1}{336}$.

b) Simplify $\frac{(3n+1)!}{(3n)!}$.

Solution: Observe that $(3n+1)! = (3n+1)(3n)!$. Hence $\frac{(3n+1)!}{(3n)!} = \frac{(3n+1)(3n)!}{(3n)!} = 3n + 1$.

6.1.2 Series

Sequences give rise to the notion of *series*. When consecutive terms of a sequence are summed this forms a series. We represent series with *summation notation* represented by the Greek letter sigma

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + ... + a_n.$$

Here the n is the *upper limit of summation* and, similarly 1 is the *lower limit of summation*. If n is a finite number this is called a *finite series*. We can also have infinite series

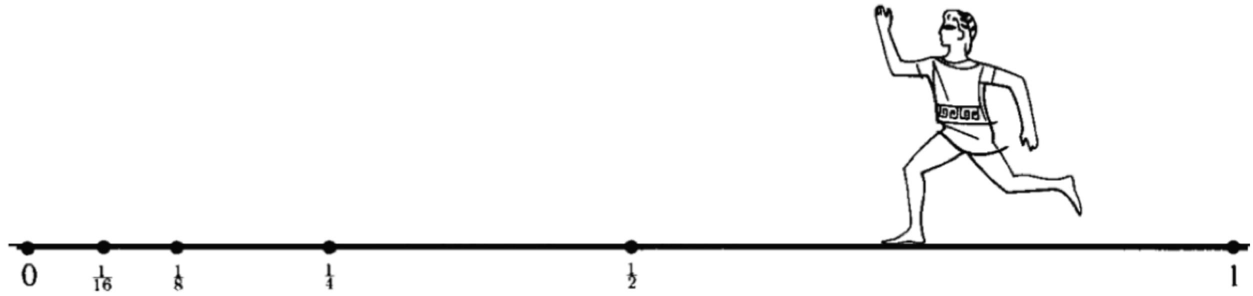
$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + ... + a_n + ...$$

One might think "Wouldn't adding up infinitely many numbers just result in adding up to infinity?" Greek Philosopher Zeno of Elea (495-435 B.C.) introduced the *racecourse paradox* [**?**] also known as *Zeno's paradox*. This paradox states this:

A runner can never reach the end of a racecourse because he must cover half of any distance before he covers the whole. That is to say, having covered the first half he still has the second half before him. When half of this is covered, one-fourth yet remains. When half of this one-fourth is covered, there remains one-eighth, and so on.

Zeno was, of course, thinking about an idealized situation in which the runner is thought of as a particle moving from one end of a line segment to the other. In particular, we can think of the runner starting at the point 1 and running to the point 0 as in the image below.

Figure 1: The Racecourse Paradox



The positions $\frac{1}{2}$, $\frac{1}{4}$, etc. indicate the fraction of the course yet to be covered. These fractions will partition the course into infinitely many smaller portions of the course. Each portion of the course would take some positive amount of time to cover, and it seems reasonable to think that the time required to cover the whole course must be the sum of the total of the sum of the time intervals to cover each portion of the course. Zeno argued that a sum of infinitely many positive numbers (in this case time intervals) cannot possibly add up to be a *finite* number. This seems paradoxical since we *know* from our physical intuition that it *is* possible to finish the race in a finite amount of time. Moreover, Zeno's racecourse paradox implicitly asserts that *any* motion is impossible. In particular, the racecourse paradox can be applied to any distance. Hence, you can never travel any amount of distance in a finite time and, therefore, motion cannot even begin.

Zeno's assertion that an infinite amount of positive quantities cannot have a finite sum was rejected 2000 years later when the theory of infinite series was created, and we will see many examples of infinite series that do add up to finite values.

Let us first take a look at and example of a finite series.

Example 1: Summing the First n Positive Integers

Suppose we wish to add up $\sum_{k=1}^{1000} k = 1+2+3+...+1000$. This would be a quite cumbersome task if we did not put some thought into it first. Suppose that, in general, we wish to add up the first n integers, call this sum $\mathcal{S}_n$. First we can think of adding them up from smallest to largest:

$$\mathcal{S}_n = 1+2+3+...+(n-1)+n.$$

Since addition is commutative, this is the same as adding them up from largest to smallest:

$$\mathcal{S}_n = n+(n-1)+...+2+1.$$

Adding these equations and grouping appropriately, we see that

$$2\mathcal{S}_n = (n+1)+(n-1+2)+(n-2+3)+...+(2+n-1)+(1+n) = (n+1)+(n+1)+...+(n+1) = n(n+1).$$

That is $2\mathcal{S}_n = n(n+1)$ and hence

$$\mathcal{S}_n = \frac{n(n+1)}{2}. \tag{6.1}$$

This is a general formula for the the sum of the first n positive integers. To see that this works, try it for $n = 5$. We can compute $1+2+3+4+5 = 15$ with ease normally. Computing $\mathcal{S}_5$ with equation 5.2 we have

$$\mathcal{S}_5 = \frac{5\cdot 6}{2} = 15.$$

Therefore we see that our formula agrees with our usual way of addition, so we can have confidence in the formula. We could easily use this formula to compute $\mathcal{S}_{1000}$ or in fact *any* positive integer.

Example 2

Let c be some constant number. We wish to compute $\sum_{i=1}^{n} ca_i$.

Solution: Writing out the sum in regular notation looks like $ca_1 + ca_2 + ... + ca_n = c(a_1 + a_2 + ... + a_n)$ where we have factored out c because it appears in each term. But observe that this is then equal to $c\sum_{i=1}^{n} a_i$. That is,

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i.$$

We can similarly prove the following properties of sums.

Properties of Sums

Let c be a constant. Then,

1. $\sum_{i=1}^{n} c = cn$,

2. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$.

3. $\sum_{i=1}^{n} a_i \pm b_i = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$, *Remark.* Note that we have not claimed that these work for infinite series.

Given an infinite sequence $a_1, a_2, ..., a_n, ...$ there is an associated infinite series $\sum_{i=1}^{\infty} a_i$. However, there is also an associated finite series $\sum_{i=1}^{n} a_i$ called the n^{th} *partial sum*. This will become useful later. As a final example, let us look at an example of an infinite series.

Example 3: Finding the Sum of an Infinite Series

Consider the series $\sum_{i=1}^{\infty} \frac{3}{10^i}$. The third partial sum is

$$\sum_{i=1}^{3} \frac{3}{10^i} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} = 0.3 + 0.03 + 0.003.$$

From this it should be clear that

$$\sum_{i=1}^{\infty} \frac{3}{10^i} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} = 0.3 + 0.03 + 0.003 + 0.0003 + 0.00003 + ... = 0.33333... = \frac{1}{3}.$$

This is, in particular, an example of an infinite series that adds up to a finite value, which Zeno had claimed was impossible.

6.2 Arithmetic Sequences and Partial Sums

6.2.1 Introduction

If a_n is a sequence, the difference between consecutive terms is $a_{n+1} - a_n$. If there exists a constant τ such that $\tau = a_{n+1} - a_n$ for all n then a_n is called an *arithmetic sequence*. The number τ is called the *common difference* of the arithmetic sequence.

Example 1

Determine whether the sequences are arithmetic. If so, find the common difference.

a) $4, 7, 10, 13, 16, ..$

Solution: The difference between each consecutive term is 3 so the sequence is arithmetic.

b) $80, 40, 20, 10, 5...$

Solution: This is clearly not arithmetic since $40 - 80 = -40$ and $10 - 20 = -10$ so the difference between consecutive terms does *not* remain constant.

Suppose that a_n is an arithmetic sequence. Then there exists a constant τ such that

$$a_2 - a_1 = \tau \implies a_2 = \tau + a_1. \tag{6.2}$$

But then we must also have $a_3 - a_2 = \tau$ also. In particular,

$$a_3 = \tau + a_2. \tag{6.3}$$

Substituting the value for a_2 from equation 5.2 into the equation for a_3 from equation 5.3 we have

$$a_3 = \tau + (\tau + a_1) = 2\tau + a_2.$$

Continuing this process we obtain the following recursive definition of an arithmetic sequence.

Recursive Definition of an Arithmetic Sequence

The n^{th} term of an arithmetic sequence whose common difference is τ has the form

$$a_n = (n-1)\tau + a_1. \tag{6.4}$$

If we define $c = a_1 - \tau$ we obtain another useful form of the above equation.

$$a_n = \tau n + c. \tag{6.5}$$

Recall that finding the n^{th} term of a sequence is difficult in general. However, this recursive formula is efficacious for finding the n^{th} term of an arithmetic sequence. The following examples illustrate this.

Example 2

Find a formula for the n^{th} term of the sequence $10, 5, 0, -5, -10, ...$

Solution: We can label $a_1 = 10$, $a_2 = 5$, etc. and notice that the common difference $\tau = -5$. Therefore the value for c in equation 5.5 is $c = 10 - (-5) = 15$ and, therefore, we have

$$a_n = -5n + 15$$

Example 3

Given that $a_1 = \frac{5}{8}$ and $a_{k+1} = a_k - \frac{1}{8}$, find the common difference of this arithmetic sequence and write the n^{th} term of the sequence.

Solution: We know that $a_{k+1} = a_k - \frac{1}{8}$ but this implies that $a_{k+1} - a_k = -\frac{1}{8}$ which is the consecutive difference between two terms. Since this is assumed to be an arithmetic sequence, this implies that $\tau = -\frac{1}{8}$. Therefore we know $c = a_1 - \tau = \frac{5}{8} + \frac{1}{8} = \frac{3}{4}$. Hence by equation 5.5

$$a_n = -\frac{n}{8} + \frac{3}{4}.$$

Remark. Think about how difficult it would be to come up with the n^{th} term for that sequence without equation 5.5.

6.2.2 The Sum of a Finite Arithmetic Sequence

We can derive the formula for a finite arithmetic sequence a_n by using equation 5.5 and the same method as in example 1 of section 5.1.2. The equation we would obtain is

The Sum of a Finite Arithmetic Sequence

The sum of a finite arithmetic sequence with n terms is given by

$$\mathcal{S}_n = \frac{n}{2}(a_1 + a_n).$$

This is a convenient way to sum arithmetic sequences. The formula is straightforward to use, so let us do a nontrivial example.

Example 1

The sum of the first 20 terms of an arithmetic a_k sequence with a common difference of 3 is 650. Find the first term.

Solution: We know by the equation for the sum of a finite arithmetic sequence that

$$\mathcal{S}_{20} = 10(a_1 + a_{20}) = 650$$

Dividing through by 10 we obtain

$$a_1 + a_{20} = 65.$$

If we can find a_{20} then we know the value of a_1. Fortunately, we can use the recursive definition of an arithmetic sequence given by equation 5.4. This equation gives us $a_{20} = a_1 + (19)(3)$. Substituting this value for a_{20} into the above equation yields

$$2a_1 + 57 = 65.$$

Therefore $a_1 = 4$.

Example 2

Determine the seating capacity of an auditorium with 36 rows if there are 15 seats in the first row, 18 seats in the second row, 21 seats in the third row, and so on.

Solution: Observe that the number of seats in each row is forming an arithmetic sequence a_k with common difference $\tau = 3$. There are a total of 36 rows so we need to compute the sum of the 36 terms of this sequence. To use the formula we for the sum of an arithmetic sequence we need to know a_1 and a_{36}. We already know $a_1 = 15$ and we can find a_{36} using the recursive definition of an arithmetic sequence. This gives $a_{36} = 15 + (35)(3) = 120$. Now we use the equation for the sum of an arithmetic sequence to compute

$$\mathcal{S}_{36} = \sum_{k=1}^{36} a_k = \frac{36}{2}(15 + 120) = 2430.$$

6.3 Geometric Sequences and Series

6.3.1 Introduction

Another important type of sequence is the *geometric sequence* which is defined as follows:

Definition 6.3. A sequence a_n is *geometric* if the ratios of consecutive terms is constant. In particular, there exists a number $r \neq 0$ such that

$$\frac{a_{n+1}}{a_n} = r$$

for all n. The number r is called the *common ratio.*

Example 1

An example of a geometric sequence is given by

$$a_n = \frac{1}{2^n}.$$

The first four terms (starting with $n = 1$) are $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$,... Observe that the common ratio is $\frac{1}{2}$ since $\frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}, \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{2}$, etc. Alternatively, we can use the general formula for a_n to see that

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}} = \frac{2^n}{2^{n+1}} = \frac{1}{2}.$$

Suppose now that we have a geometric series a_n with common ratio r. Then we have that $\frac{a_2}{a_1} = r$ so that

$$a_2 = a_1 r.$$

Similarly, $\frac{a_3}{a_2} = r$ so that $a_3 = a_2 r = a_1 r^2$ by the previous equation. Continuing in this fashion we see that

$$a_n = a_1 r^{n-1}$$

Therefore we can define a geometric sequence recursively as follows

The nth Term of a Geometric Sequence

The n^{th} term of a geometric sequence has the form

$$a_n = a_1 r^{n-1} \tag{6.6}$$

where r is the common ratio of consecutive terms of the sequence.

We can use this recursive definition of a geometric sequence to easily find the n^{th} term of a geometric sequence similar to as we did for arithmetic sequences.

Example 2

Find the n^{th} term of the geometric sequence with the given information.

$a_1 = 48$, $a_{k+1} = -\frac{1}{2}a_k$.

Solution: Using $a_{k+1} = -\frac{1}{2}a_k$ we know that $r = \frac{a_{k+1}}{a_k} = -\frac{1}{2}$ is the common ratio. Substituting this value of r and the given value for a_1 into equation 5.6 obtains

$$a_n = 48(-\frac{1}{2})^{n-1}.$$

Here is a useful formula for the sum of a finite geometric series.

Proposition 6.1. The Sum of a Finite Geometric Sequence *The sum of a finite geometric sequence a_k with common ratio $r \neq 1$ is given by*

$$\mathcal{S}_n = \sum_{k=1}^{n} a_1 r^{k-1} = a_1\left(\frac{1 - r^n}{1 - r}\right).$$

Proof. Using the recursive definition of a geometric series, we have

$$\mathcal{S}_n = \sum_{k=1}^{n} a_1 r^{k-1}.$$

Multiplying this expression for $\mathcal{S}_n$ by r we have

$$r\mathcal{S}_n = r\sum_{k=1}^{n} a_1 r^{k-1} = \sum_{k=1}^{n} a_1 r^k$$

where we have used property (2) of the properties of sums to move r inside the sum. Subtracting these two equations yields

$$\mathcal{S}_n - r\mathcal{S}_n = a_1 - a_1 r^n.$$

Therefore since $r \neq 1$ we have

$$\mathcal{S}_n(1-r) = a_1(1-r^n) \implies \mathcal{S}_n = a_1\left(\frac{1-r^n}{1-r}\right)$$

□

We can extend the sum of a finite geometric series to the sum of an *infinite* geometric series provided that $|r| < 1$. For if this is the case, then

$$\lim_{n\to\infty} a_1\left(\frac{1-r^n}{1-r}\right) \longrightarrow a_1\left(\frac{1-0}{1-r}\right)$$

The notation $\lim_{n\to\infty}$ means "the limit as n approaches infinity". This means that you let n get arbitrarily large. Since $|r| < 1$, raising r^n to large powers makes it get closer and closer to zero. In particular, r^n gets arbitrarily close to zero which means that its limit is zero, so we get the expression on the right.

Remark. It is okay if you do not understand this. This sort of thing will be studied further in a calculus course.

Summarizing this, we have

The Sum of an Infinite Geometric Series

If $|r| < 1$ then the geometric series $\sum\limits_{k=1}^{\infty} a_k$ has the sum

$$\mathcal{S} = \sum_{k=1}^{\infty} a_1 r^{k-1} = \frac{a_1}{1-r}.$$

Remark. It is essential that $|r| < 1$. If $|r| \geq 1$ then the sum is infinite.

Example 3: The Sum of a Finite Geometric Series

a) Find the value of $\sum\limits_{k=1}^{12} 16(\frac{1}{2})^{k-1}$.

Solution: Substuting the values $a_1 = 16, r = \frac{1}{2}$, and $n = 12$ into the equation for the sum of a finite geometric series gives

$$16\left(\frac{1-(\frac{1}{2})^{12}}{1-\frac{1}{2}}\right) = \frac{4095}{128}.$$

b) Find the value of the $\sum\limits_{k=0}^{15} 2(\frac{4}{3})^k$.

Solution: Observe that this sum is equivalent to $\sum_{k=1}^{16} 2(\frac{4}{3})^{k-1}$ (this is called re-indexing). Then substituting the values $a_1 = 2, r = \frac{4}{3}$, and $k = 16$ into the equation for the sum of a finite geometric series gives

$$\sum_{k=1}^{16} 2(\frac{4}{3})^{k-1} = 2\left(\frac{1-(\frac{4}{3})^{16}}{1-\frac{4}{3}}\right) = \frac{8503841150}{14348907} \approx 592.6.$$

Example 4

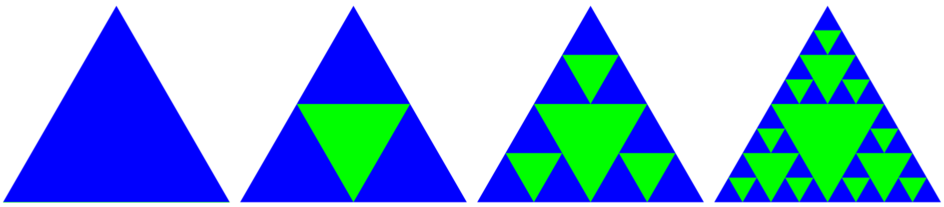
Find the sum of the series $\sum_{k=1}^{\infty} -10\left(\frac{2}{10}\right)^{k-1}$.

Solution: Since $r < 1$ the sum exists as a finite value. Substituting $a_1 = -10$, $r = \frac{2}{10}$ into the equation for the sum of an infinite geometric series gives

$$\sum_{k=1}^{\infty} -10\left(\frac{2}{10}\right)^{k-1} = \frac{-10}{1-\frac{2}{10}} = -\frac{25}{2}.$$

Example 5: The Sierpinski Triangle

A *fractal* is a geometric figure that consists of a pattern that is repeated infinitely on a smaller and smaller scale. A well-known fractal is called the *Sierpinski Triangle.* In the first stage, the midpoints of the three sides of an equilateral triangle are used to create the vertices of a new triangle, which is then removed, leaving three triangles. In the second stage, this is repeated for each of the remaining triangles. This process continues for the remaining triangles after the n^{th} stage. Note that each of the remaining triangle are similar to the original triangle. Assume that the length of each side of the original triangle is one unit. The first few stages are shown below, where the green triangles are the portions that have been removed and the blue triangles are the remaining portions.



a) Write a formula that describes the side length of the triangles that will be generated in the n^{th} stage.

Solution: Let ℓ_n be the sequence whose values are the lengths of the sides of the triangles generated in the n^{th} stage. Observe that $\ell_1 = 1$, $\ell_2 = \frac{1}{2}$, $\ell_3 = \frac{1}{4}, \ell_4 = \frac{1}{8}$ etc. We recognize this to be the sequence from Example 1 which happens to be a geometric sequence. In particular, we have

$$\ell_n = \frac{1}{2^{n-1}}$$

is the length of the sides in the n^{th} stage.

b) Write a formula for the area of each blue triangle that will be generated in the n^{th} stage.

Since the original triangle was an equilateral triangle, and each triangle at every stage is similar to the original triangle, it follows that all of the triangles formed will also be equilateral. We can use proposition **??** to find an explicit formula for the area of an equilateral triangle. In particular, in an equilateral triangle all of the sides are equal length, namely ℓ_n at stage n, and all of the angles are 60°. Hence using proposition **??** we obtain the area of a triangle at stage n is

$$\mathcal{A}_n = \frac{\ell_n^2}{2}\sin 60^\circ = \frac{\ell_n^2}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{\ell_n^2\sqrt{3}}{4}.$$

Moreover, using exponent rules to simplify we have $\ell_n^2 = \left(\frac{1}{2^{n-1}}\right)^2 = \frac{4}{4^n}$ and substituting this value into the formula for $\mathcal{A}_n$ gives

$$\mathcal{A}_n = \frac{\sqrt{3}}{4^n}.$$

c) Determine the total area of all of the blue triangles at step n.

The total area of the blue triangles will be the sum of the areas of all of the blue triangles. The question is how many blue triangles are there. At each stage, every blue triangle splits into 3 more blue triangles in the next step. It is easily seen then that the number of blue triangles at step n is

$$\mathcal{N}_n = 3^{n-1}.$$

Therefore the total area of all of the blue triangles combined will be

$$\mathcal{N}_n\mathcal{A}_n = \frac{3^{n-1}\sqrt{3}}{4^n} = \frac{\sqrt{3}}{4}\left(\frac{3}{4}\right)^{n-1}.$$

An interesting observation is that as $n \to \infty$ this expression goes to zero. So the total area of the blue triangle is zero after infinitely many steps.