3.10.2 First find F_Y: $F_Y(y) = \int_0^y 3t^2\,dt = y^3$.

Then $P(Y_5' > 0.75) = 1 - P(Y_5' < 0.75)$, where

$$P(Y_5' < 0.75) = \int_0^{0.75} \frac{6!}{(5-1)!\,(6-5)!}(y^3)^{5-1}(1-y^3)^{6-5}\,3y^2\,dy$$

$$= \int_0^{0.75} \frac{6!}{4!}(y^3)^4(1-y^3)3y^2\,dy = \int_0^{0.75} \frac{6!}{4!}(y^3)^4(1-y^3)3y^2\,dy$$

$$= \int_0^{0.75} 90(y^{14})(1-y^3)\,dy = 90\left[\frac{y^{15}}{15} - \frac{y^{18}}{18}\right]_0^{0.75} = 0.052,$$

so $P(Y_5' > 0.75) = 1 - P(Y_5' < 0.75) = 1 - 0.052 = 0.948$

3.11.2 The probability that $X = x$ and $Y = y$ is the probability of y 4's on the first two rolls and $x - y$ rolls on the last four rolls. These events are independent, so

$$p_{X,Y}(x, y) = \binom{2}{y}\left(\frac{1}{6}\right)^{y}\left(\frac{5}{6}\right)^{2-y}\binom{4}{x-y}\left(\frac{1}{6}\right)^{x-y}\left(\frac{5}{6}\right)^{4-x+y} \quad \text{for } y \le x$$

Then $$p_{Y|x}(y) = \frac{p_{X,Y}(x,y)}{p_X(x)} = \frac{\binom{2}{y}\left(\frac{1}{6}\right)^{y}\left(\frac{5}{6}\right)^{2-y}\binom{4}{x-y}\left(\frac{1}{6}\right)^{x-y}\left(\frac{5}{6}\right)^{4-x+y}}{\binom{6}{x}\left(\frac{1}{6}\right)^{x}\left(\frac{5}{6}\right)^{6-x}} = \frac{\binom{2}{y}\binom{4}{x-y}}{\binom{6}{x}},$$

$0 \le y \le \min(2, x)$, which we recognize as a hypergeometric distribution.

3.11.12 (a) $f_X(x) = \int_x^\infty 2e^{-x}e^{-y}\,dy = 2e^{-2x},\ x > 0$

So $P(X < 1) = \int_0^1 2e^{-2x}dx = 1 - e^{-2} = 1 - 0.135 = 0.865$

Also, $P(X < 1, Y < 1) = \int_0^1\int_0^x 2e^{-(x+y)}dydx$

$= \int_0^1 2e^{-x}\left[-e^{-y}\right]_0^x dx = \int_0^1 (2e^{-x} - 2e^{-2x})dx$

$= -2e^{-x} + e^{-2x}\Big|_0^1 = 0.400$

Then the conditional probability is $\dfrac{0.400}{0.865} = 0.462$

(b) $P(Y < 1 \mid X = 1) = 0$, since the joint pdf is defined with y always larger than x.

(c) $f_{Y|x}(y) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{2e^{-(x+y)}}{2e^{-2x}} = e^{x}e^{-y},\ x < y$

3.12.6 $M_Y(t) = E(e^{tY}) = \int_0^1 e^{ty} y\, dy + \int_1^2 e^{ty}(2-y)\, dy$

$$= \left(\frac{1}{t}y - \frac{1}{t^2}\right)e^{ty}\Big|_0^1 + \frac{2}{t}e^{ty}\Big|_1^2 - \left(\frac{1}{t}y - \frac{1}{t^2}\right)e^{ty}\Big|_1^2$$

$$= \left(\frac{1}{t} - \frac{1}{t^2}\right)e^t - \left(-\frac{1}{t^2}\right) + \frac{2}{t}e^{2t} - \frac{2}{t}e^t - \left(\frac{2}{t} - \frac{1}{t^2}\right)e^{2t} + \left(\frac{1}{t} - \frac{1}{t^2}\right)e^t$$

$$= \frac{1}{t^2} + \frac{1}{t^2}e^{2t} - \frac{2}{t^2}e^t = \frac{1}{t^2}(e^t - 1)^2$$