MATH 2433-006
Exam III
April 30, 2010

ANSWERS

1. Find $\mathbf{a}+\mathbf{b}, 2\mathbf{a}+3\mathbf{b}, |\mathbf{a}|$, and $|\mathbf{a}-\mathbf{b}|$ for

$$\mathbf{a}=\mathbf{i}+2\mathbf{j}-3\mathbf{k},\ \mathbf{b}=-2\mathbf{i}-\mathbf{j}+5\mathbf{k}$$

ANSWER

$$\begin{aligned}\mathbf{a}+\mathbf{b}&=-\mathbf{i}+\mathbf{j}+2\mathbf{k}\\ 2\mathbf{a}+3\mathbf{b}&=-4\mathbf{i}+\mathbf{j}+9\mathbf{k}\\ |\mathbf{a}|&=\sqrt{14}\\ |\mathbf{a}-\mathbf{b}|&=\sqrt{82}\end{aligned}$$

2. Determine whether the given vectors are orthogonal, parallel, or neither.

(a) $\mathbf{a}=\langle -5,3,7\rangle,\ \mathbf{b}=\langle 6,-8,2\rangle$

(b) $\mathbf{a}=2\mathbf{i}+6\mathbf{j}-4\mathbf{k},\ \mathbf{b}=-3\mathbf{i}-9\mathbf{j}+6\mathbf{k}$

ANSWER

(a) $\mathbf{a}\cdot\mathbf{b}=-40\neq 0$ so the vectors are not orthogonal. They are not scalar multiples of each other so they are not parallel.

(b) Since $\mathbf{a}=-\frac{2}{3}\mathbf{b}$, the vectors are parallel.

3. Find the scalar and vector projections of $\mathbf{b}$ along $\mathbf{a}$.

$$\mathbf{a}=\langle 3,6,-2\rangle,\ \mathbf{b}=\langle 1,2,3\rangle$$

ANSWER The scalar projection is

$$\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$$

The vector projection is

$$\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$$

Since $\mathbf{a} \cdot \mathbf{b} = 3 \cdot 1 + 6 \cdot 2 + -2 \cdot 3 = 9$ and $|\mathbf{a}| = \sqrt{9 + 36 + 4} = 7$, the scalar projection is $\frac{9}{7}$ and the vector projection is $\frac{9}{49}\langle 3, 6, -2\rangle$.

4. If $\mathbf{a} = \langle 1, 2, 1\rangle,\ \mathbf{b} = \langle 0, 1, 3\rangle$ find $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$

ANSWER $\mathbf{a} \times \mathbf{b} = \langle 5, -3, 1\rangle$. Since $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$, $\mathbf{b} \times \mathbf{a} = \langle -5, 3, -1\rangle$.

5. Find a nonzero vector orthogonal to the plane through $P(1, 0, 0)$, $Q(0, 2, 0)$, and $R(0, 0, 3)$

ANSWER The plane contains the vectors $\vec{PQ} = \langle 0-1, 2-0, 0-0\rangle = \langle -1, 2, 0\rangle$ and $\vec{PR} = \langle 0-1, 0-0, 3-0\rangle = \langle -1, 0, 3\rangle$. So the vector $\langle -1, 2, 0\rangle \times \langle -1, 0, 3\rangle = \langle 6, 3, 2\rangle$ is orthogonal to the plane.

6. Find parametric equations and symmetric equations for the line through the points $(0, \frac{1}{2}, 1)$ and $(2, 1, -3)$

ANSWER The line passes through $(0, \frac{1}{2}, 1)$ in the direction $\langle 2-0, 1-\frac{1}{2}, -3-1\rangle = \langle 2, \frac{1}{2}, -4\rangle$ so the parametric equations are

$$\begin{aligned} x &= 0 + 2t \\ y &= \frac{1}{2} + \frac{1}{2}t \\ z &= 1 - 4t \end{aligned}$$

and the symmetric equations are

$$\frac{x}{2} = \frac{y - \frac{1}{2}}{\frac{1}{2}} = \frac{z - 1}{-4}$$

7. Find an equation of the plane through the point $(6, 3, 2)$ and perpendicular to the vector $\langle -2, 1, 5\rangle$

ANSWER The equation is

$$-2(x - 6) + 1(y - 3) + 5(z - 2) = 0$$

8. Find the derivative of the vector function $\mathbf{r}(t) = \langle t\sin t, t^2, t\cos 2t\rangle$

ANSWER

$$\langle t\cos t + \sin t, 2t, -2t\sin 2t + \cos 2t\rangle$$

9. Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point.

$$x = 1 + 2\sqrt{t},\ y = t^3 - t,\ z = t^3 + t;\ (3, 0, 2)$$

ANSWER Consider the corresponding vector function

$$\mathbf{r}(t) = \langle 1 + 2\sqrt{t}, t^3 - t, t^3 + t\rangle$$

Then

$$\mathbf{r}'(t) = \langle 2\frac{1}{2}t^{-\frac{1}{2}}, 3t^2 - 1, 3t^2 + 1\rangle$$

The number t for which $\mathbf{r}(t) = \langle 3, 0, 2\rangle$ has $t^3 - t = 0$, so $t = -1, 0, 1$ and since also $t^3 + t = 2$, $t = 1$. Then

$$\mathbf{r}'(1) = \langle 1, 2, 4\rangle$$

and the line through $(3, 0, 2)$ in the direction $\langle 1, 2, 4\rangle$ has parametric equations

$$\begin{aligned} x &= 3 + t \\ y &= 2t \\ z &= 2 + 4t \end{aligned}$$

10. Set up the integral to find the length of the curve

$$\mathbf{r}(t) = \mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k},\ 0 \le t \le 1$$

ANSWER The length is given by

$$\int_0^1 |\mathbf{r}'(t)| dt$$

Since

$$\mathbf{r}'(t) = \langle 0, 2t, 3t^2\rangle$$

we have

$$|\mathbf{r}'(t)| = \sqrt{4t^2 + 9t^4}$$

and the length is given by

$$\int_0^1 \sqrt{4t^2 + 9t^4} dt$$