Assignment #1 Due 2-1-05
p 26 2-7, 2-15 p 46 3.19

2.7 $C \subseteq \mathbb{Z}_2^{12}$ $d(C) = 3$

(a) $\exists$ codeword of odd weight and even weight

Proof Let $u, v \in C$, $d(u,v) = 3$. We may assume that bits where u, v differ are first three:

$$u = u_1 u_2 u_3\ u_4 — u_{12} \qquad v = v_1 v_2 v_3\ v_4 — v_{12}$$

$u_i \neq v_i$ $1 \le i \le 3$; $u_i = v_i$ $4 \le i \le 12$

If wt $(u_1 u_2 u_3)$ = odd, wt $(v_1 v_2 v_3)$ = even and vice versa.

If wt $(u_1 u_2 u_3)$ = odd and wt $(u_4 — u_{12})$ = wt $(v_4 — v_{12})$ = even then wt $(u_1 — u_{12})$ = odd + even = odd and

wt $(v_1 — v_{12})$ = even + even = even.

If wt $(u_1 u_2 u_3)$ = even and wt $(u_4 — u_{12})$ = wt $(v_4 — v_{12})$ = even then wt $(u_1 — u_{12})$ = ~~odd~~ even + even = even and

wt $(v_1 — v_{12})$ = odd + even = odd.

If wt $(u_1 u_2 u_3)$ = even and wt $(u_4 — u_{12})$ = wt $(v_4 — v_{12})$ = ~~odd~~ then wt $(u_1 — u_{12})$ = even + ~~odd~~ = ~~odd~~ and

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wt $(v_1 — v_{12})$ = ~~odd~~ even + odd = odd.

In every case, wt (u) and wt (v) have different parity.

b) $K \subseteq \mathbb{Z}_2^{13}$ where $x = x_1 — x_{13} \in K$ is defined by $x_1 — x_{12} \in C$ and $x_{13} = 1$ if wt $(x_1 — x_{12})$ odd, $x_{13} = 0$ if wt $(x_1 — x_{12})$ even. Then $d(K) = 4$.

Proof Let $u, v \in C$ as in (a), and assume wt (u) odd, wt (v) even. Let $\bar{u} = u_1 — u_{12} u_{13}$, $\bar{v} = v_1 — v_{12} v_{13}$ be the corresponding element of K.

wt(u) odd $\Rightarrow u_{13}=1$; wt(v) even $\Rightarrow v_{13}=0$.
Thus $d(\bar{u},\bar{v})=4$ (differ in places 1, 2, 3, 13). Thus $d(K)\le 4$. Suppose $\exists\, x=x_1 — x_{13}$, $y=y_1 — y_{13}\in K$ with $d(x,y)\le 3$. If $x_{13}\ne y_{13}$, then $x'=x_1 — x_{12}$ and $y'=y_1 — y_{12}\in C$ and $d(x',y')\le 2$, contrary to $d(C)=3$. If $x_{13}=y_{13}$ then $x',y'\in C$, $d(x',y')\le 3$ and either both have odd or both have even weight. Since $d(C)=3$, $d(x',y')=3$, but then by (a) their weights are of different parity. Thus no such x,y and $d(x,y)\ge 4$ $\forall x,y\in K$ so $d(K)=4$

(c) $d(C)=3=2e+1\Rightarrow$ C corrects single errors
$d(K)=4=2\cdot 1+1+1\Rightarrow K$ corrects single errors and detects $1+1=2$ errors

2.15 Let C_i denote C shortened at i:

$$C_i=\{u_1 - u_{i-1}u_{i+1} - - u_n \mid u_1 - u_n\in C \text{ and } u_i=0\}$$

Let $x,y\in C_i$, so $x=x_1 - x_{i-1}x_{i+1} - x_n$, $x_1 - x_n\in C$
$y=y_1 - - y_{i-1}y_{i+1} - y_n$, $y_1 - y_n\in C$
and $x_i=y_i=0$. Thus $d(x,y)=d(x_1 - x_n, y_1 - y_n)\ge d(C)$ if $x\ne y$.
Thus $d(C_i)\ge d(C)$ [Note: could have $C_i=\phi$ or $|C_i|=1$, so $d(C_i)$ not defined]

Consider $C=\{000,011\}\subseteq \mathbb{Z}_2^3$ $|C|=2^1\Rightarrow$ rank $C=1$
$C_1=\{00,11\}\subseteq\mathbb{Z}_2^2$ $|C_1|=2^1\Rightarrow$ rank $C_1=1$
So rank C_i can equal rank C

3.19 C is a linear binary code. Either half the code words have even weights, or they are all of even weights.

Proof Let $C_e = \{ x \in C \mid wt(x) \text{ even} \}$

$C_o = \{ x \in C \mid wt(x) \text{ odd} \}$.

Then $C = C_o \cup C_e$ and $C_o \cap C_e = \emptyset$, so $|C| = |C_o| + |C_e|$. If $C = C_e$ then all words are of even weight. If not, $\exists\, u_0 \in C_o$. Consider the function $f: C_e \longrightarrow C_o$ by $f(u) = u + u_0$ (note $u + u_0 \in C$ because C linear and u even $\Longrightarrow$ $u + u_0$ odd), and the function $g: C_o \longrightarrow C_e$ by $g(v) = v + u_0$. Then $g f(u) = u + u_0 + u_0 = u + 0 = u$ and $g f(v) = v + u_0 + u_0 = v + 0 = v$ so g, f are inverse bijections between C_o & C_e. Thus $|C_o| = |C_e|$ and $|C| = 2|C_e|$ so $|C_e| = \frac{1}{2}|C|$. So if not all words are even, half are.