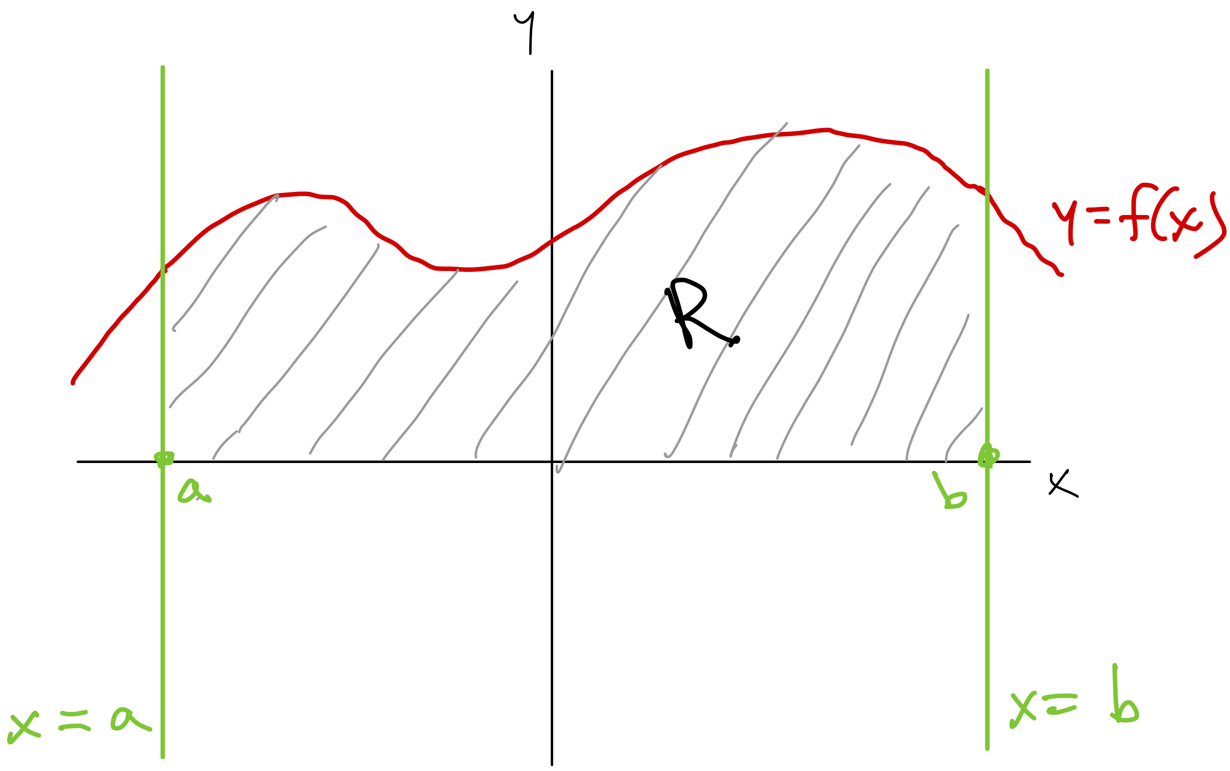
General Problem (4.1)

Suppose that $f(x) \geq 0$ for $a \leq x \leq b$. How can we approximate the area of the region R bounded by $y = f(x)$, $y = 0$, $x = a$, $x = b$?



This region R can be described by inequalities :

$$R : \begin{cases} a \leq x \leq b \\ 0 \leq y \leq f(x) \end{cases}$$

Q: Can a region R have negative area?

A: No! The area of a region (if it exists) is never, ever negative!!

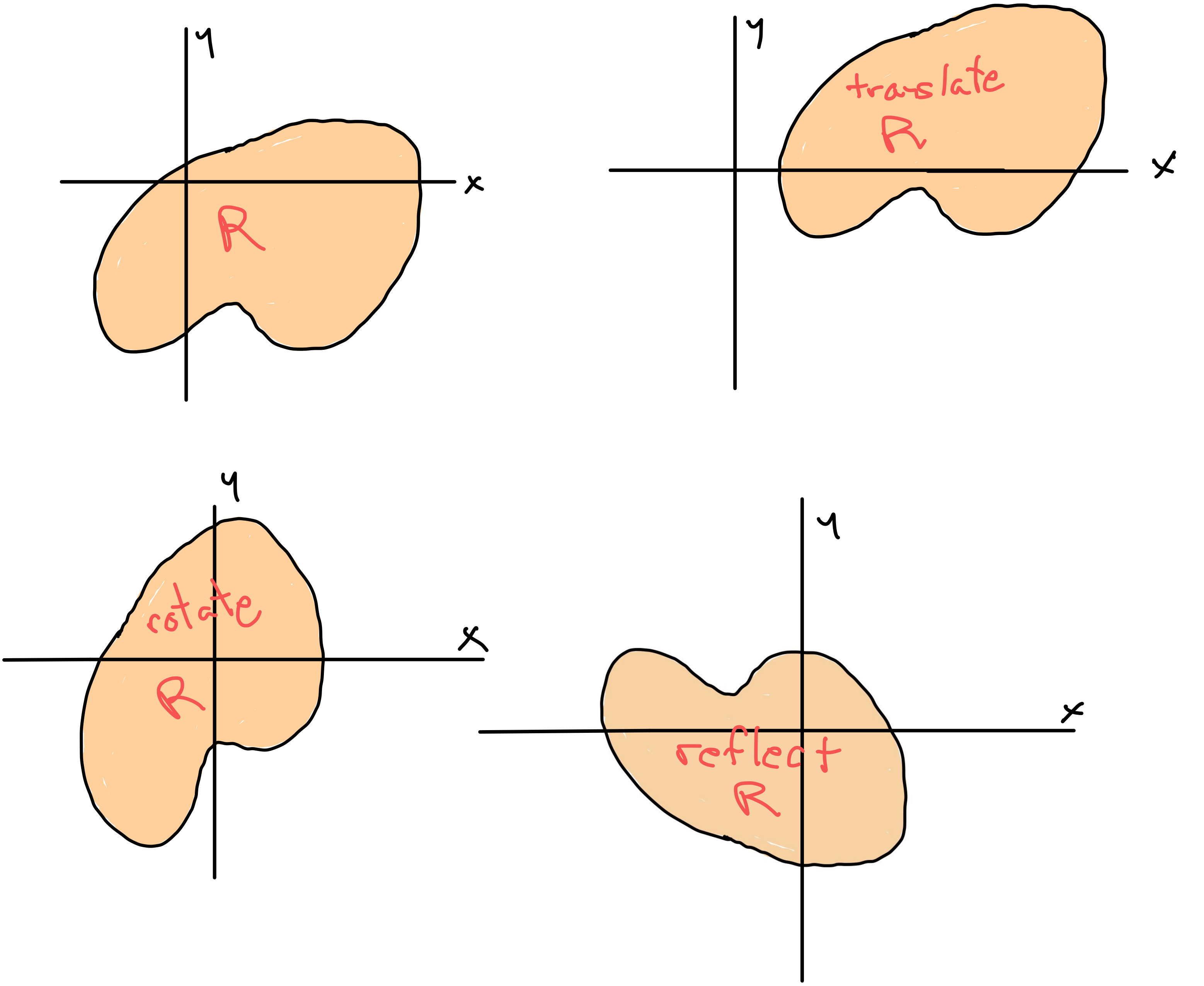
Q: So area is always positive?

A: Yes - well, not quite, because sometimes the area of R might equal 0, and techically 0 is not positive. The correct statement is area(R) is non-negative (technically).

Example A line segment has area 0. Think of this as a rectangle with width zero.

More about area:

Area and volume are really geometry concepts and not calculus concepts.



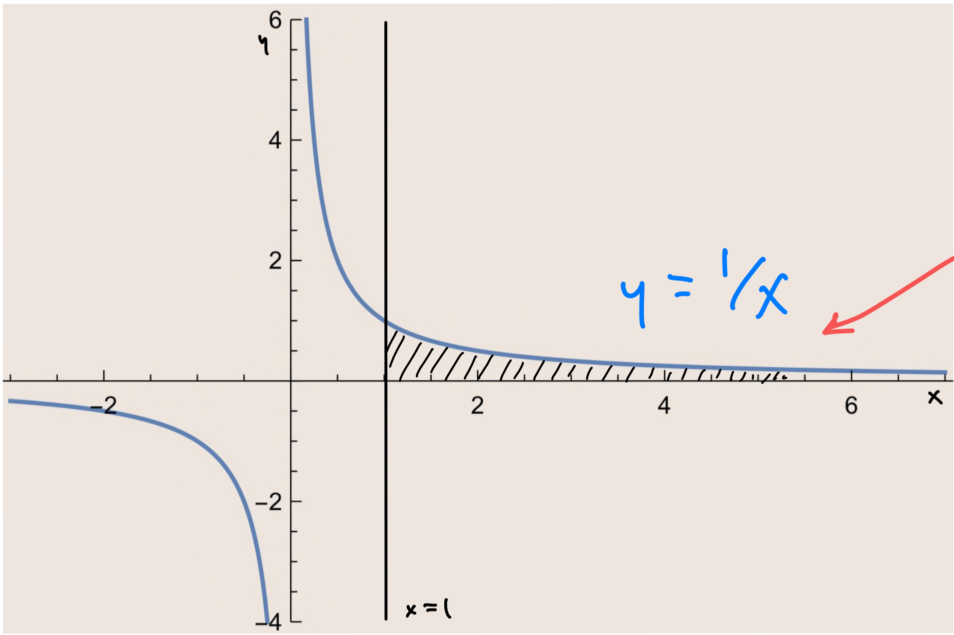
Moving R in any of these ways does not change its area.

("reflect" $\equiv$ "take mirror image")

So where does calculus come into area?

A: Calculus can be used to actually calculate the area of certain special types of regions.

Area can be non-intuitive at times

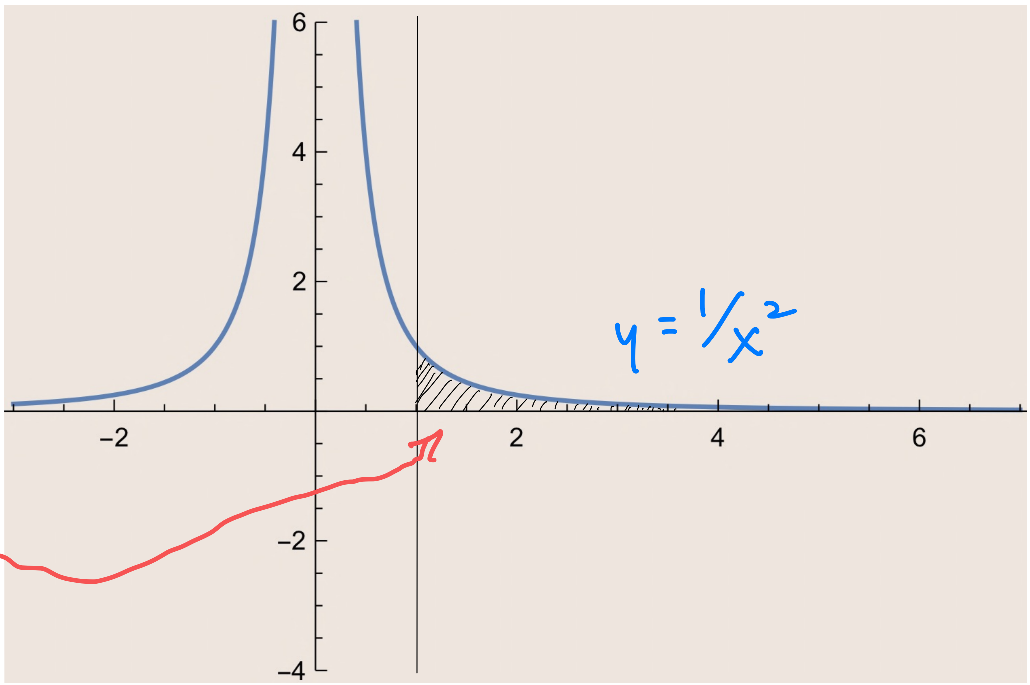


This region has infinite area.

$$\begin{cases} 1 \le x < +\infty \\ 0 \le y \le 1/x \end{cases}$$

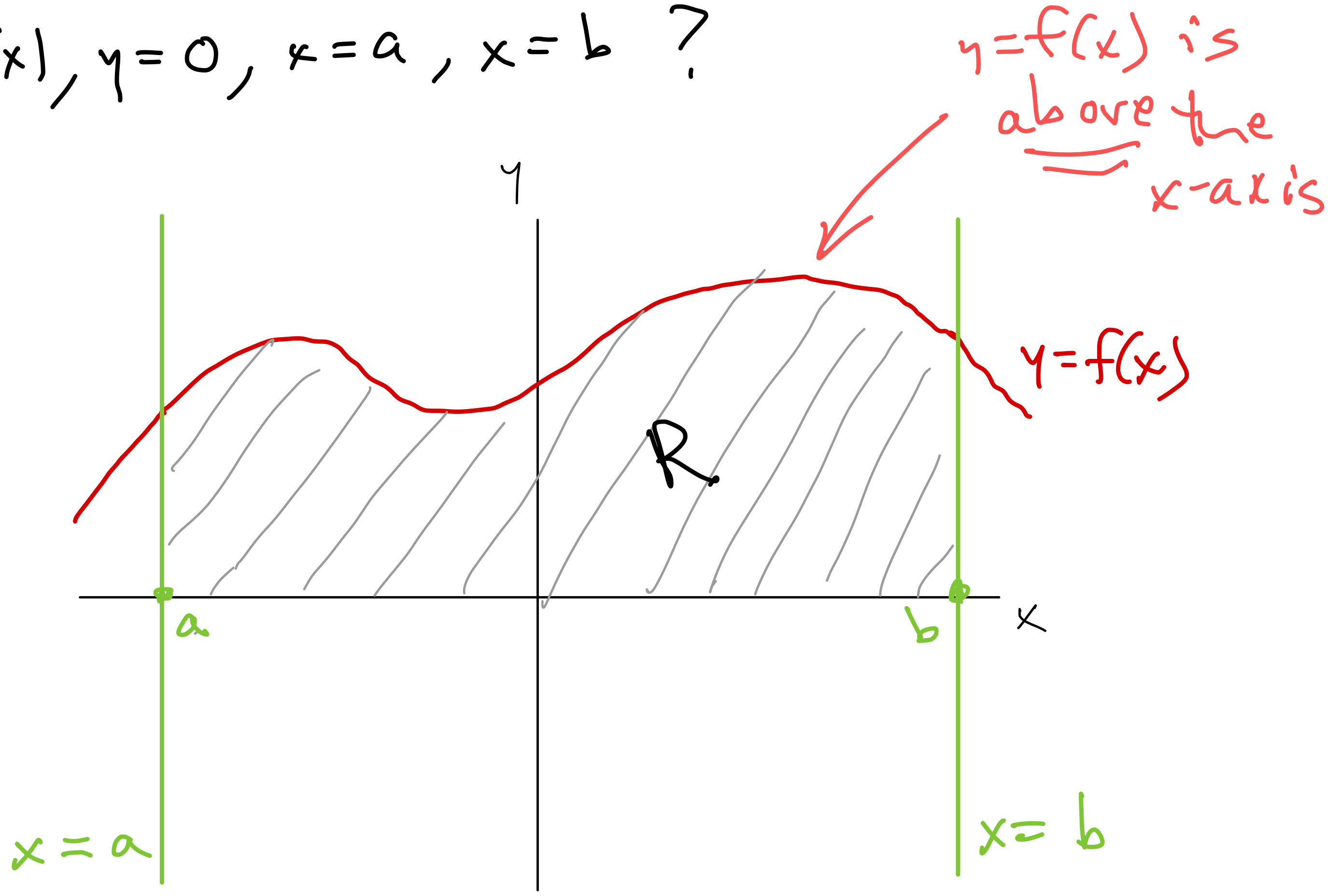
This region has area 1

$$\begin{cases} 1 \le x < \infty \\ 0 \le y \le \frac{1}{x^2} \end{cases}$$



General Problem (section 4.1 in Stewart)

Suppose that $f(x) \geq 0$ for $a \leq x \leq b$. How can we approximate the area of the region R bounded by $y = f(x)$, $y = 0$, $x = a$, $x = b$?



This region R can be described by inequalities:

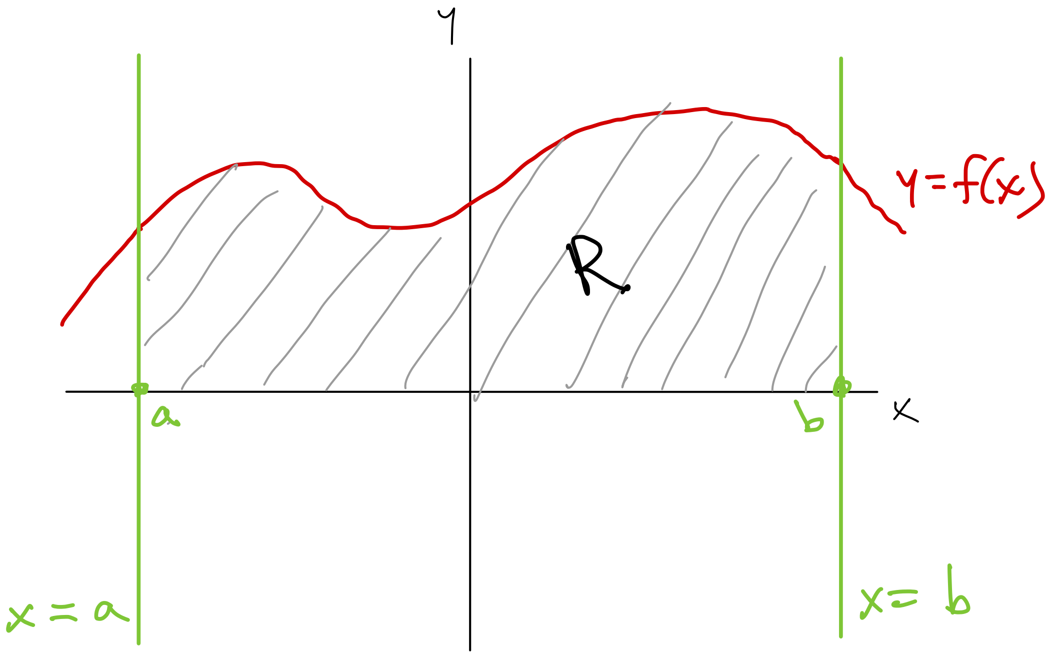
left side

right side

$$R: \begin{cases} a \leq x \leq b \\ 0 \leq y \leq f(x) \end{cases}$$

bottom side

top side



Approximate area(R) as follows:

① Subdivide the interval $I=[a,b]$ into N subintervals $I_1, \ldots, I_N$ with length $\Delta x = (b-a)/N$

② In each interval I_k choose a point x_k^*.

③ Erect a rectangle R_k above the interval I_k with height $f(x_k^*) \geq 0$.

④ Adding the areas of these rectangles R_k gives an approximation to area(R).

⑤ Repeat the process for larger values of N to improve the approximation.

The approximation can be written:

$$\text{area}(R) \approx \text{area}(R_1) + \text{area}(R_2) + \cdots + \text{area}(R_N)$$

$$= \underbrace{f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_N^*)\Delta x}_{N \text{ terms added together.}}$$

$$= \sum_{k=1}^{N} f(x_k^*)\Delta x$$

This is called a <u>Riemann Sum</u> for $f(x)$ over the interval $[a,b]$

$$\boxed{\begin{aligned}&\text{Area}(R_k)\\ &\quad = f(x_k^*)\,\Delta x\end{aligned}}$$

<u>Sigma Notation:</u>

$$\sum_{k=1}^{4} \frac{k^2}{3} = \frac{1^2}{3} + \frac{2^2}{3} + \frac{3^2}{3} + \frac{4^2}{3}$$

$$\sum_{k=3}^{6} \frac{k^2}{3} = \frac{3^2}{3} + \frac{4^2}{3} + \frac{5^2}{3} + \frac{6^2}{3}$$

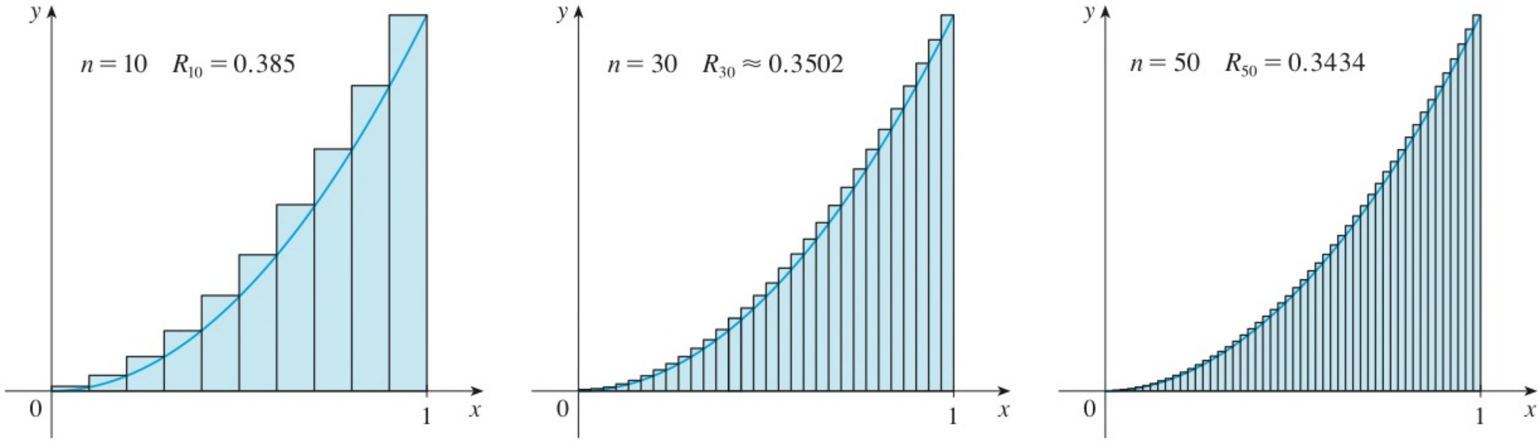


FIGURE 8 Right endpoints produce upper sums because $f(x) = x^2$ is increasing.

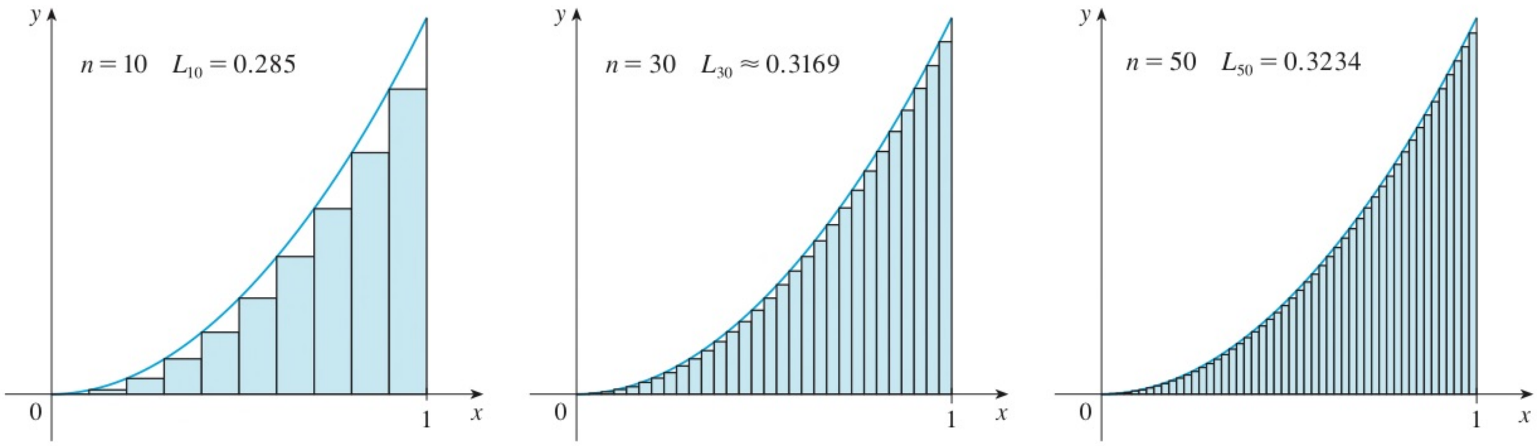


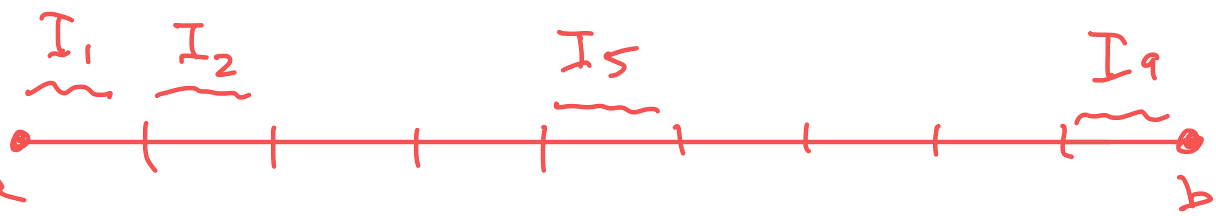
FIGURE 9 Left endpoints produce lower sums because $f(x) = x^2$ is increasing.

Stewart : page 297

Let's apply the idea of Examples 1 and 2 to the more general region S of Figure 1

Note x_k^* can be any point in the I_k interval.
but we often make special choices :

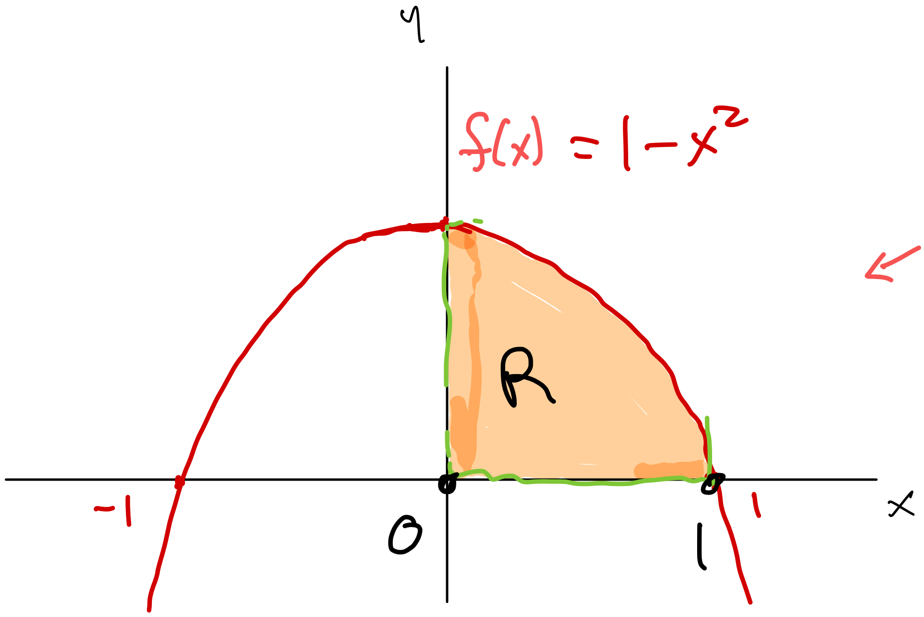
- x_k^* = left endpoint of I_k
- x_k^* = right endpoint of I_k
- x_k^* = midpoint of I_k



picture for N=9

Problem

Find the area of the shaded region.



$[a,b] = [0,1]$

$R: \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1-x^2 \end{cases}$

area(R) = ??

$x_k^* = \frac{k-1}{N}$

$\Delta x = \frac{1}{N}$

left Riemann sums

$$\text{area}(R) = \sum_{k=1}^{N} f\left(\frac{k-1}{N}\right)\Delta x = \sum_{k=1}^{N}\left(1-\left(\frac{k-1}{N}\right)^2\right)\frac{1}{N}$$

Table

N	$\sum_{k=1}^{N}\left(1-\left(\frac{k-1}{N}\right)^2\right)\frac{1}{N}$
2	.875
10	.715
50	.6766
100	.67165
1000	.6671665
10^4	.6671665
10^{10}	.6666666717

These Numbers look like they have a limit of 2/3

Theorem If $f(x)$ is continuous over $[a,b]$ then $\lim\limits_{N\to\infty} \sum_{k=1}^{N} f(x_k^*)\Delta x$ exists.

Riemann Sums

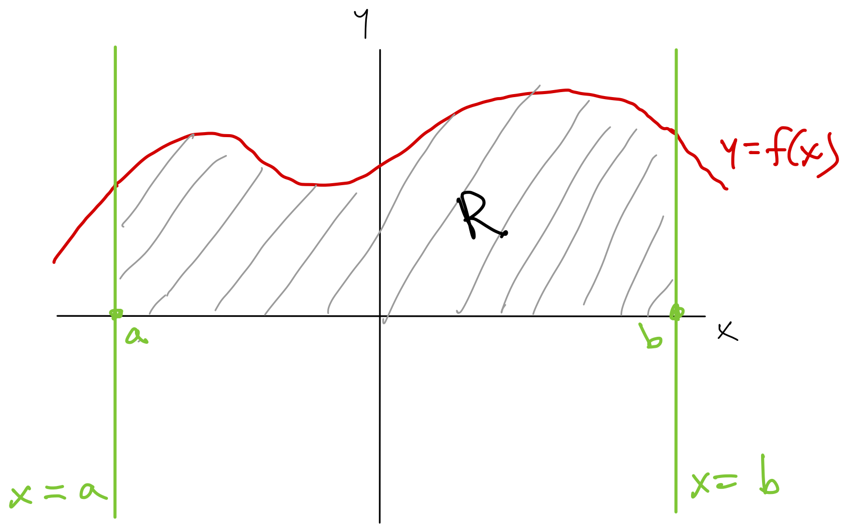
We write

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{k=1}^{\infty} f(x_k^*)\Delta x$$

and call this the integral of $f(x)$ over the interval $[a,b]$.

From this we can write:

Area(R) $= \int_a^b f(x)\,dx$



At best Riemann Sums are very tedious to calculate*, but they are very important because:

- They can provide very close approximations to the value of $\int_a^b f(x)\,dx$ when a precise calculation is not possible.
- They govern how to understand the meaning of $\int_a^b f(x)\,dx$ in applications. For example they show that the area of the region $R: a \le x \le b,\ 0 \le y \le f(x)$ (where $f(x) \ge 0$) equals $\int_a^b f(x)\,dx$.
- They can used to a few important basic algebraic properties of integrals.

* However, computers can be programmed to quickly calculate Riemann Sums.

example

$\frac{1}{x^2}$ not defined when $x=0$.

$$\int_{-1}^{1} \frac{1}{x^2}\, dx$$

$f(x) = \frac{1}{x^2}$ is not continuous on $[-1, 1]$

Make sure that $[a,b]$ is contained in domain(f)