True or False?

① $\int \pi f(x)\,dx = \pi \int f(x)\,dx$. True.

② $\int x f(x)\,dx = x \int f(x)\,dx$. False. (very false!)

Linearity Principle:

If a and b are constants then

$$\int a f(x) + b g(x)\,dx = a \int f(x)\,dx + b \int g(x)\,dx$$

take $b=0$ $\quad \int a f(x)\,dx = a \int f(x)\,dx$

take $a=1=b$ $\quad \int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$

note: You can only do this if a is constant with respect to x! (If a has any x's in it, then the equality is not true.)

Substitution (section 4.5)

Other than the basic rules for anti-differentiating the basic elementary functions, by far the most useful tool for calculating integrals comes from the chain rule. The "anti-chain rule" yields a technique called the "method of substitution".

Here's Stewart's description (from page 341):

4 The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

In this description $u = g(x)$, so that $\frac{du}{dx} = g'(x)$, which can be written as $du = g'(x)\,dx$. We summarize by writing:

$$\begin{cases} u = g(x) \\ du = g'(x)\,dx \end{cases}$$

Now Rule $\boxed{4}$ just amounts to substituting these equations into the integral:

$$\int \underbrace{f(g(x))}_{\text{substitute } u=g(x) \text{ to get } f(u)} \underbrace{g'(x)\,dx}_{\text{substitute } du=g'(x)\,dx} = \int f(u)\,du$$

Substitution Method

If $\begin{cases} u = g(x) \\ du = g'(x)\,dx \end{cases}$

then $\int f(g(x))\, g'(x)\,dx = \int f(u)\,du$

The strategy is to replace $\int f(g(x))g'(x)\,dx$ with the integral $\int f(u)\,du$ which (hopefully) is easier to calculate.

examples

① $\int \cos(x^3)\,3x^2\,dx$.

If we choose $u = x^3$ then $\frac{du}{dx} = 3x^2$. So we substitute $\begin{cases} u = x^3 \\ du = 3x^2\,dx \end{cases}$ to get

$$\int \cos(x^3)\,3x^2\,dx = \int \cos(u)\,du \overset{①}{=} \sin(u) + C \overset{②}{=} \sin(x^3) + C$$

where ① is a known formula, and in ② we need to remember to substitute back in to express the final answer as a function of x. (The final answer should never have any u's in it.)

② $\int \cos(x^7)\,x^6\,dx$.

This time we'll use $\begin{cases} u = x^7 \\ du = 7x^6\,dx \end{cases}$ and get

$$\int \cos(x^7)\,x^6\,dx \overset{③}{=} \frac{1}{7}\int \cos(x^7)\,7x^6\,dx = \frac{1}{7}\int \cos u\,du$$

$$= \frac{1}{7}\sin(u) + C = \frac{1}{7}\sin(x^7) + C$$

(Notice that linearity was used at ③.)

example ② continued ...

We found that $\boxed{\int \cos(x^7)\, x^6\, dx = \frac{1}{7}\sin(x^7) + C}$.

Check the answer by differentiating:

$$\frac{d}{dx}\left[\frac{1}{7}\sin(x^7)\right] = \frac{1}{7}\cos(x^7)\cdot\frac{d}{dx}\left[x^7\right] = \frac{1}{7}\cos(x^7)\, 7x^6$$

$$= \cos(x^7)\, x^6$$

(which is correct).

This step used the Chain Rule!

Look at examples 2–5 that Stewart discusses on pages 342–343 to start to get a feel for when to use the method of substitution, and how to choose a substitution that works. Experience is the best way to get a handle on this.

Always keep in mind that "integration is hard".

example ③ There is no substitution that will allow you to calculate the integral

$$\int \cos(x^7)\, dx$$

(In fact this integral is impossible to work in closed form — we'll discuss this more at a later time.)

Here's the idea:

Suppose that $F(x)$ is an antiderivative for $f(x)$.
This means that
$$F'(x) = f(x) \quad \text{and} \quad \int f(x)\,dx = F(x) + C .$$
Now consider the derivative of $F(g(x))$:
$$\frac{d}{dx}\left[F(g(x))\right] = F'(g(x))\,g'(x) = f(g(x))\,g'(x)$$
chain rule

In terms of integrals this says
$$\int f(g(x))\,g'(x)\,dx = F(g(x)) + C$$
$$= F(u) + C = \int f(u)\,du$$
where $u = g(x)$.

This shows how the method of substitution comes from the chain rule.

example

$$\begin{cases} u = \cos x \\ du = -\sin x \, dx \end{cases}$$

$$(-1)\int (\cos^2 x + 1)^7 (-1)\sin x \, dx$$

a degree 14 polynomial

$$= (-1)\int (u^2+1)^7 \, du = -\int (u^2+1)^7 \, du$$

Now "just" expand $(u^2+1)^7$ and work the integral,,,,

$$(u^2+1)^7 = 1 + 7u^2 + 21u^4 + 35u^6 + 35u^8 + 21u^{10} + 7u^{12} + u^{14}$$

So $\int (u^2+1)^7 \, du =$

$$u + \frac{7}{3}u^3 + \frac{21u^5}{5} + \frac{35u^7}{7} + \frac{35}{9}u^9 + \frac{21}{11}u^{11} + \frac{7}{13}u^{13} + \frac{u^{15}}{15} + C$$

And $\int (\cos^2 x + 1)^7 \sin x \, dx =$

$$-\Big\{\cos x + \frac{7}{3}\cos^3 x + \frac{21}{5}\cos^5(x) + \frac{35}{7}\cos^7 x + \frac{35}{9}\cos^9 x + \frac{21}{11}\cos^{11}(x) + \frac{7}{13}\cos^{13}(x) + \frac{1}{15}\cos^{15}(x)\Big\} + C$$

Conclusion Integration is hard ?