Substitution Method If $\begin{cases} u = g(x) \\ du = g'(x)\,dx \end{cases}$

then $\int f(g(x))\, g'(x)\,dx = \int f(u)\,du$

TRY: Make the substitution $\begin{cases} u = x^5 - 1 \\ du = 5x^4\,dx \end{cases}$

to rewrite each as $\int f(u)\,du$.

① $\int \underbrace{\sqrt{x^5-1}}_{u}\ \underbrace{5x^4\,dx}_{du} = \int \sqrt{u}\,du = \int u^{1/2}\,du$

② $\frac{1}{5}\int \sqrt{x^5-1}\ {}^{5}\,x^4\,dx = \frac{1}{5}\int \sqrt{u}\,du$

③ $\frac{1}{5}\int \underbrace{(x^5-1)}_{u}\underbrace{5x^4\,dx}_{du} = \frac{1}{5}\int u\,du$

④ $\int (x^5-1)^{100} x^4\,dx = \frac{1}{5}\int u^{100}\,du$

⑤ $\int \frac{x^4}{(x^5-1)^{100}}\,dx = \frac{1}{5}\int \frac{1}{u^{100}}\,du$

$= \frac{1}{5}\int u^{-100}\,du$

Stewart, Section 4.5

7–30 Evaluate the indefinite integral

These are the substitutions that should get you started.

7. $\int x\sqrt{1-x^2}\,dx$ $\begin{cases} u=x^2 \\ du=2x\,dx \end{cases}$

8. $\int x^2\sin(x^3)\,dx$ $u=x^3$

9. $\int (1-2x)^9\,dx$ $\begin{cases} u=1-2x \\ du=-2\,dx \end{cases}$

10. $\int \sin t\sqrt{1+\cos t}\,dt$ $u=\cos t$

11. $\int \sin(2\theta/3)\,d\theta$ $u=2\theta/3$

12. $\int \sec^2 2\theta\,d\theta$ $u=2\theta$

13. $\int \sec 3t\tan 3t\,dt$ $u=3t$

14. $\int y^2(4-y^3)^{2/3}\,dy$ $u=4-y^3$

15. $\int \cos(1+5t)\,dt$ $u=1+5t$

16. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$ $u=\sqrt{x}$

17. $\int \sec^2\theta\tan^3\theta\,d\theta$ $u=\tan\theta$

18. $\int \sin x\sin(\cos x)\,dx$ $u=\cos x$

19. $\int (x^2+1)(x^3+3x)^4\,dx$ $\begin{cases} u=x^3+3x \\ du=3(x^2+1)\,dx \end{cases}$

20. $\int x\sqrt{x+2}\,dx$ $u=x+2$

21. $\int \frac{a+bx^2}{\sqrt{3ax+bx^3}}\,dx$ $\begin{cases} u=3ax+bx^3 \\ du=3(a+bx^2)\,dx \end{cases}$

22. $\int \frac{\cos(\pi/x)}{x^2}\,dx$ $\begin{cases} u=\pi/x \\ du=-\pi/x^2\,dx \end{cases}$

23. $\int \frac{z^2}{\sqrt[3]{1+z^3}}\,dz$ $u=1+z^3$

24. $\int \frac{dt}{\cos^2 t\sqrt{1+\tan t}}$ $u=\tan t$

25. $\int \sqrt{\cot x}\csc^2 x\,dx$ $u=\cot x$

26. $\int \frac{\sec^2 x}{\tan^2 x}\,dx$ $u=\tan x$

27. $\int \sec^3 x\tan x\,dx$ $\begin{cases} u=\sec x \\ du=\sec x\tan x\,dx \end{cases}$

28. $\int x^2\sqrt{2+x}\,dx$ $u=2+x$

29. $\int x(2x+5)^8\,dx$ $\begin{cases} u=2x+5 \\ du=2dx \\ x=(u-5)/2 \end{cases}$

30. $\int x^3\sqrt{x^2+1}\,dx$ $\begin{cases} u=x^2+1 \\ du=2x\,dx \\ x^2=u-1 \end{cases}$

Substitution Method If $\begin{cases} u = g(x) \\ du = g'(x)\,dx \end{cases}$

then $\int f(g(x))\, g'(x)\,dx = \int f(u)\,du$

The strategy is to replace $\int f(g(x))g'(x)dx$ with $\int f(u)\,du$ which (hopefully) is easier to calculate.

Examples:

substitute: $\begin{cases} u = \sqrt{t} \\ du = \frac{1}{2}\frac{1}{\sqrt{t}}\,dt \end{cases}$

$\begin{cases} u = t^{1/2} \\ du = \frac{1}{2}t^{-1/2}dt = \frac{1}{2}\frac{1}{\sqrt{t}}\,dt \end{cases}$

① $2\int \frac{1}{2}\frac{\cos(\sqrt{t})}{\sqrt{t}}\,dt$

$= 2\int \cos(u)\,du = 2\sin(u) + C$

$= 2\sin(\sqrt{t}) + C$

A substitution is not as obvious here but try:

$\begin{cases} u = 2 + x^2 \\ du = 2x\,dx \end{cases}$

and observe that

$x^2 = u - 2$

② $\int x^3\sqrt{2+x^2}\,dx$

$= \frac{1}{2}\int x^2\sqrt{2+x^2}\,2x\,dx$

$= \frac{1}{2}\int (u-2)\sqrt{u}\,du = \frac{1}{2}\int (u-2)u^{1/2}\,du$

$= \frac{1}{2}\int u^{3/2} - 2u^{1/2}\,du = \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2}\right) + C$

$= \frac{1}{5}(2+x^2)^{5/2} - \frac{2}{3}(2+x^2)^{3/2} + C$

$u \cdot u^{1/2} = u^{1+1/2} = u^{3/2}$

Now use the "anti-power rule"

example (from last class)

substitute
$$\begin{cases} u = \cos x \\ du = -\sin x \, dx \end{cases}$$

$$\int (\cos^2 x + 1)^7 \sin x \, dx$$

a degree 14 polynomial

$$= (-1)\int (u^2+1)^7 \, du = -\int (u^2+1)^7 \, du$$

Now "just" expand $(u^2+1)^7$ and work the integral, Like this:

$$(u^2+1)^7 = 1 + 7u^2 + 21u^4 + 35u^6 + 35u^8 + 21u^{10} + 7u^{12} + u^{14}$$

So $\int (u^2+1)^7 \, du =$

$$u + \frac{7}{3}u^3 + \frac{21u^5}{5} + \frac{35u^7}{7} + \frac{35}{9}u^9 + \frac{21}{11}u^{11} + \frac{7}{13}u^{13} + \frac{u^{15}}{15} + C$$

And $\int (\cos^2 x + 1)^7 \sin x \, dx =$

$$-\left[\cos x + \frac{7}{3}\cos^3 x + \frac{21}{5}\cos^5(x) + \frac{35}{7}\cos^7 x + \frac{35}{9}\cos^9 x + \frac{21}{11}\cos^{11}(x) + \frac{7}{13}\cos^{13}(x) + \frac{1}{15}\cos^{15}(x)\right] + C$$

Conclusion Integration is hard.

Do you agree?

note: No easier way to work this integral.

Working definite integrals via substitution requires some care regarding the limits of integration!

Example: $\int_2^5 \frac{1}{\sqrt{x-1}}\,dx$

method 1: Include the limits in the substitution data.

$$\int_2^5 \frac{1}{\sqrt{x-1}}\,dx = \int_{1=u(2)}^{4=u(5)} \frac{1}{\sqrt{u}}\,du \qquad \begin{cases} u = x-1 & u(2)=1 \\ du = dx & u(5)=4 \end{cases}$$

$$= \int_1^4 u^{-1/2}\,du = 2u^{1/2}\Big|_{u=1}^{4} = 2\sqrt{4} - 2\sqrt{1} = 2$$

method 2: First work the indefinite integral

$$\int \frac{1}{\sqrt{x-1}}\,dx = \int \frac{1}{\sqrt{u}}\,du = \int u^{-1/2}\,du \qquad \begin{cases} u = x-1 \\ du = dx \end{cases}$$

$$= 2u^{1/2} + C = 2\sqrt{x-1} + C$$

So $\int_2^5 \frac{1}{\sqrt{x-1}}\,dx = 2\sqrt{x-1}\Big|_{x=2}^{5} = 2$

method 3: Clearly indicate the limits' variable.

$$\int_2^5 \frac{1}{\sqrt{x-1}}\,dx = \int_{x=2}^{5} \frac{1}{\sqrt{u}}\,du \qquad \begin{cases} u = x-1 \\ du = dx \end{cases}$$

$$= 2u^{1/2}\Big|_{x=2}^{5} = 2\sqrt{x-1}\Big|_{x=2}^{5} = 4 - 2 = 2$$

Regarding the previous example —

What you cannot write is

$$\underbrace{\int_2^5 \frac{1}{\sqrt{x-1}}\,dx}_{=2} \underset{*}{=} \underbrace{\int_2^5 \frac{1}{\sqrt{u}}\,du}_{=2\sqrt{5}-2\sqrt{2}} \qquad \begin{cases} u = x-1 \\ du = dx \end{cases}$$

Why not? Because the equation $*$ is false! Writing $\int_2^5 \frac{1}{\sqrt{u}}\,du$ indicates that we are integrating with respect to u as u goes from 2 to 5. In fact,

$$\int_2^5 \frac{1}{\sqrt{u}}\,du = 2\sqrt{u}\,\Big|_2^5 = 2\sqrt{5} - 2\sqrt{2}$$

Conclusion: If you don't want to change limits of integration then use Method 3

Average Value

Section 5.5

The average value of a list $a_1, a_2, \ldots a_N$ of N numbers equals

$$\frac{1}{N}(a_1 + a_2 + \cdots + a_N) = \frac{1}{N}\sum_{k=1}^{N} a_k = \sum_{k=1}^{N} \frac{a_k}{N}.$$

Is it possible to define the average value of a function $f(x)$ over an interval? Yes

Definition If $f(x)$ is continuous on $[a,b]$ then the average value of f on $[a,b]$ is

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

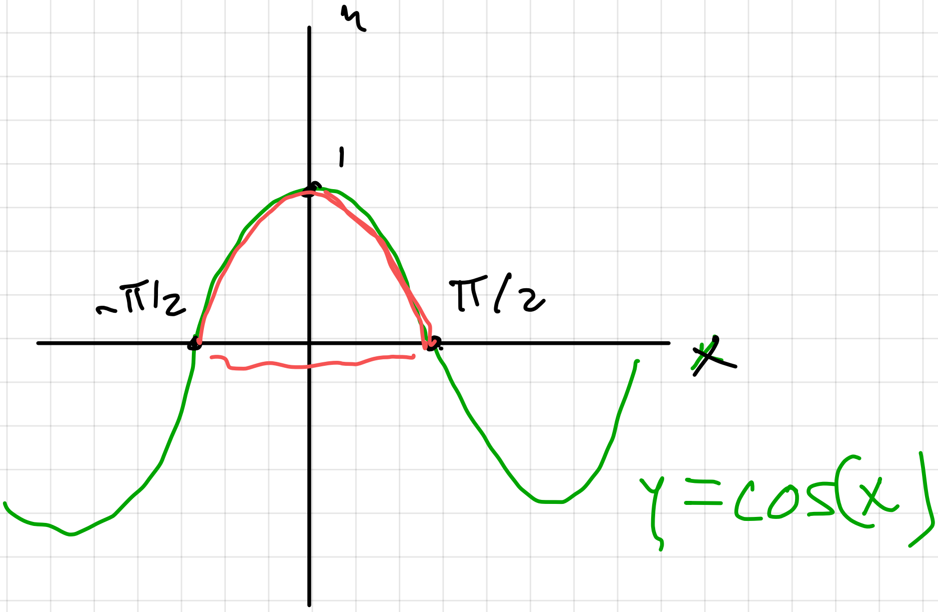
note: $b-a$ is the length of $[a,b]$.

example: Find the average value of $\cos(x)$ on the interval $[-\pi/2, \pi/2]$.

(picture on next page)

Answer Since $\frac{\pi}{2} - (-\pi/2) = \pi$, the average value is

$$\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos(x)\,dx = \frac{1}{\pi}\sin(x)\Big|_{x=-\pi/2}^{\pi/2} = \frac{1}{\pi}(1-(-1)) = \frac{2}{\pi}$$



$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ has length

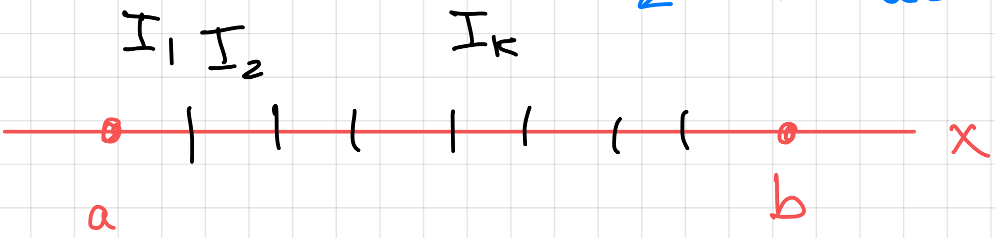
$$\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

Average Value of $\cos(x)$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $= \dfrac{2}{\pi} \approx .63662$

Explaining the definition of f_{ave}

f_{ave} = Average of $f(x)$ over $[a,b]$.

Subdivide $[a,b]$ into N subintervals.



I_k has length $\frac{b-a}{N} = \Delta x$

Pick x_k^* in the interval I_k.

Think of $f(x_k^*)$ as approximately equal to the average value of $f(x)$ over I_k. This approximation gets better as the length Δx of I_k gets smaller (that is, as N gets large). Now

$$\frac{1}{N}\left(f(x_1^*) + f(x_2^*) + \cdots + f(x_N^*)\right)$$

$$= \frac{1}{b-a}\sum_{k=1}^{N} f(x_k^*)\Delta x \xrightarrow[N\to+\infty]{} \frac{1}{b-a}\int_a^b f(x)\,dx$$

Conclude $f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$

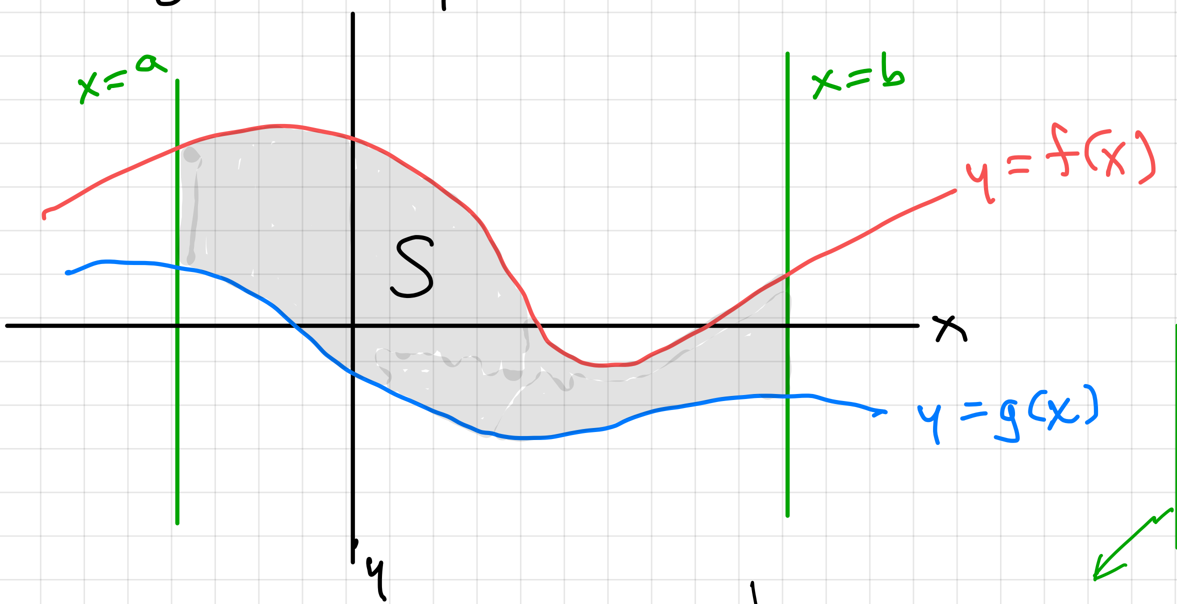
Area between curves

A region S in the xy-plane which is between the vertical lines $x = a$ and $x = b$, and is bounded above by $y = f(x)$ and below by $y = g(x)$ is called a Type I region. It is described by inequalities as

$$S : \begin{cases} a \le x \le b \\ g(x) \le y \le f(x) \end{cases}$$

note the conditions require $g(x) \le f(x)$ on the interval $[a, b]$.

and generically looks something like:



Since $a \le b$ and $f(x) \ge g(x)$ this integral is always positive

Then

$$\text{Area}(S) = \int_a^b f(x) - g(x)\, dx$$

---

observe

left side

right side

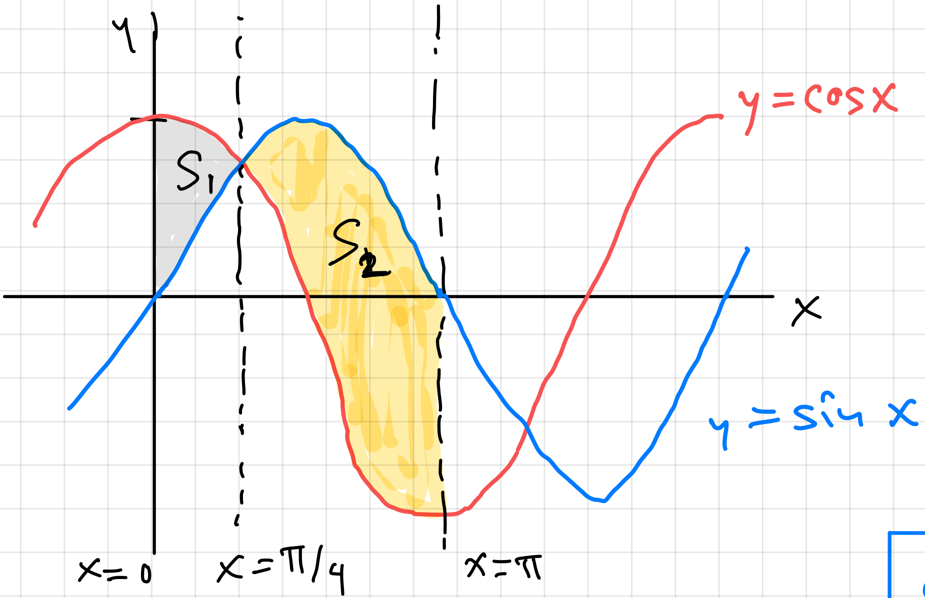
$$S : \begin{cases} a \le x \le b \\ g(x) \le y \le f(x) \end{cases}$$

bottom side

top side

Examples

① Find the area of the region S_1 between $y=\cos(x)$ and $y=\sin(x)$ for $0 \le x \le \pi/4$.



Drawing a sketch can often be helpful.

$$S_2: \begin{cases} \pi/4 \le x \le \pi \\ \cos x \le y \le \sin x \end{cases}$$

$$S_1: \begin{cases} 0 \le x \le \pi/4 \\ \sin x \le y \le \cos x \end{cases}$$

$$\text{Area}(S_1) = \int_0^{\pi/4} \cos x - \sin x \, dx = \sin x + \cos x \Big|_0^{\pi/4} =$$

$$\sin\frac{\pi}{4} + \cos\frac{\pi}{4} - \sin 0 - \cos 0 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1 = \sqrt{2} - 1$$

② Find the area of the region S_2 between $y=\cos(x)$ and $y=\sin(x)$ for $\pi/4 \le x \le \pi$.

$$\text{area}(S_2) = \int_{\pi/4}^{\pi} \sin x - \cos x \, dx = -\cos x - \sin x \Big|_{\pi/4}^{\pi} = \sqrt{2} + 1$$

③ Find the area of the region S_3 between $y=\cos(x)$ and $y=\sin(x)$ for $0 \le x \le \pi$.

$$\text{area}(S_3) = \text{area}(S_1) + \text{area}(S_2) = 2\sqrt{2}$$

watch out for +/- signs!

Comments on Example 3.

Notice that the region S_3 is <u>not</u> a Type I region because $y = \cos x$ is on top for x between 0 and $\pi/4$, but $y = \cos(x)$ is on top for x between $\pi/4$ and π. However S_3 <u>does</u> decompose into the union of S_1 and S_2, each of which do have type I.

For this reason, $\int_0^{\pi} \cos x - \sin x \, dx = -2$ does not equal the area of S_3.

However $\int_0^{\pi} |\cos x - \sin x| \, dx$ <u>does</u> represent the area of S_3, as discussed on page 360 of Stewart's book. But this observation doesn't really affect how we calculated Area(S_3) on the previous page:

For x between 0 and π, we have

$$|\cos x - \sin x| = \begin{cases} \cos x - \sin x & \text{if } 0 \le x \le \pi/4 \\ \sin x - \cos x & \text{if } \pi/4 < x \le \pi \end{cases}$$

So,

$$\int_0^{\pi} |\cos x - \sin x| \, dx = \int_0^{\pi/4} \cos x - \sin x \, dx + \int_{\pi/4}^{\pi} \sin x - \cos x \, dx$$

$$= \text{Area}(S_1) + \text{Area}(S_2)$$