True or False - What do you think?

① $\int_0^{\pi/2} \cos(x)\,dx = \int_0^{\pi/2} \cos(\Theta)\,d\Theta$ True

x is a "dummy variable"

② $\int \sec(t)\tan(t)\,dt = \int \frac{\sin(t)}{\cos^2(t)}\,dt$ True

$$\sec t \tan t = \frac{1}{\cos t} \cdot \frac{\sin t}{\cos t} = \frac{\sin t}{\cos^2 t}$$

$$\int \sec t \tan t \, dt = \sec t + C$$

$$\int \frac{\sin t}{\cos^2 t}\,dt \qquad \begin{cases} u = \cos t \\ du = -\sin t\,dt \end{cases}$$

$$= -\int \frac{1}{u^2}\,du = -\int u^{-2}\,du = -\left(\frac{1}{-1}\right) u^{-1} + C$$

$$= \frac{1}{\cos t} + C = \sec t + C$$

③ $\int x \sin(x^2)\,dx$ can be quickly solved by substitution.

True (take $u = x^2$, etc)

④ $\int x \sin(x)\,dx$ can be quickly solved by substitution.

False (see next page)

(4) $\int x\sin(x)\,dx = ?$

No substitution will get anywhere on this!

Nevertheless, if we are adventurous we could try to guess an answer.

1st attempt: try $-x\cos x$. Well:

$$\frac{d}{dx}\left[-x\cos x\right] = \frac{d}{dx}\left[-x\right]\cos(x) - x\frac{d}{dx}\left[\cos x\right] = -\cos x + x\sin x$$

because of product rule

not equal to $-x\cos x$, but

2nd attempt: try $\sin x - x\cos x$.

Here $\frac{d}{dx}\left[\sin x - x\cos x\right] = \cos x + (-\cos x + x\sin x) = x\sin x$

eureka!

Conclude: $\int x\sin(x)\,dx = \sin x - x\cos x + C$

I like to think of the process used in 4 as the "Guessing Method" for solving an integral.

<u>It works as follows:</u>

To find $\int f(x)\,dx$, guess an answer $F(x)$. Then check if it is correct by seeing if $\frac{d}{dx}[F(x)] = f(x)$.

- If it does, then $\int f(x)\,dx = F(x) + C$
- If it doesn't, then you probably haven't gotten any closer to the correct answer.!

<u>In practice:</u>

Make an educated guess for $\int f(x)\,dx$. Then work out the derivative of your guess and show that it equals $f(x)$.

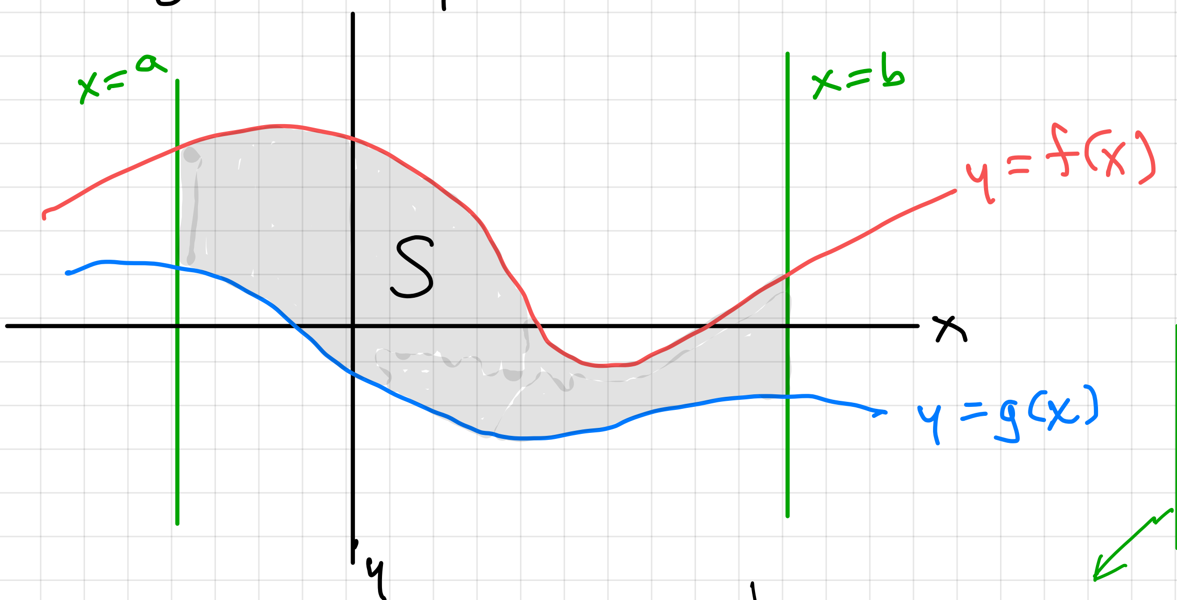
Area between curves

A region S in the xy-plane which is between the vertical lines $x=a$ and $x=b$, and is bounded above by $y=f(x)$ and below by $y=g(x)$ is called a Type I region. It is described by inequalities as

note the conditions require $g(x) \le f(x)$ on the interval $[a,b]$.

$$S : \begin{cases} a \le x \le b \\ g(x) \le y \le f(x) \end{cases}$$

and generically looks something like:



Since $a \le b$ and $f(x) \ge g(x)$ this integral is always positive

Then
$$\text{Area}(S) = \int_a^b f(x) - g(x)\,dx$$

---

observe

left side

right side

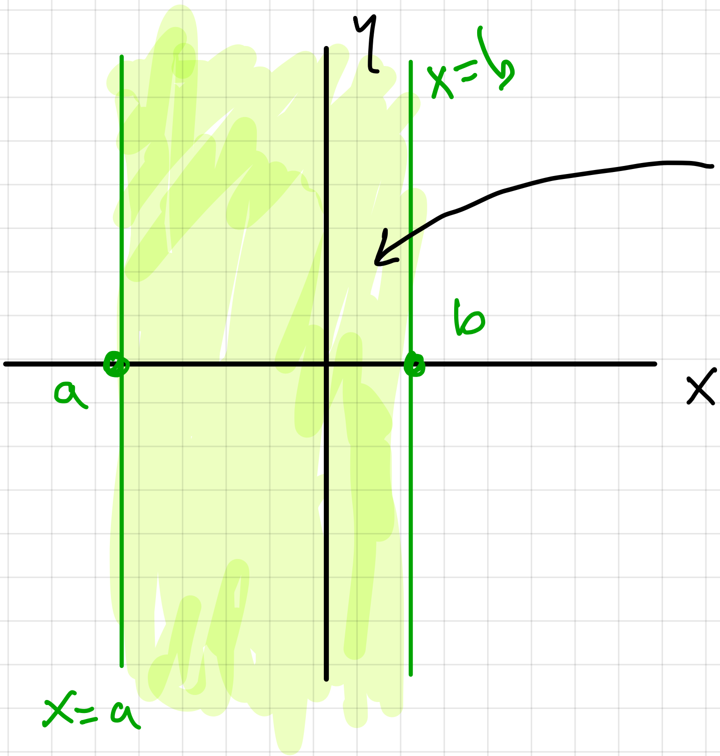
$$S : \begin{cases} a \le x \le b \\ g(x) \le y \le f(x) \end{cases}$$

top side

bottom side

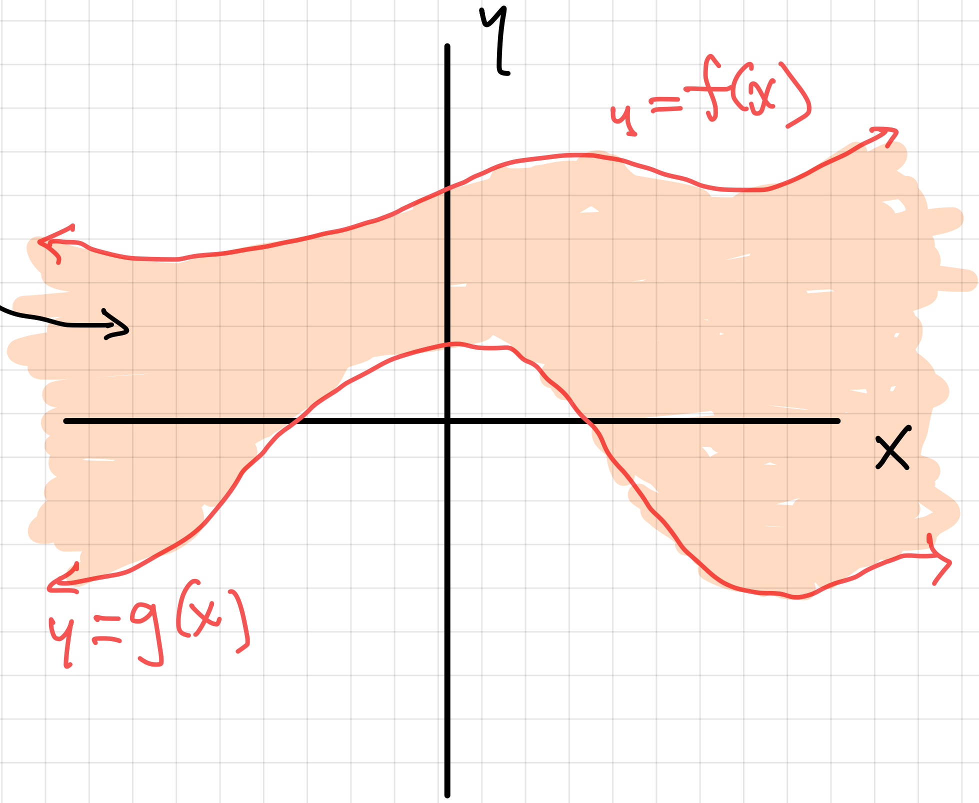
It's good to understand inequalities like these.

$$S : \begin{cases} a \le x \le b \\ g(x) \le y \le f(x) \end{cases}$$



$a \le x \le b$

$\boxed{\text{$a \le b$ means $x=b$ is to the right of $x=b$}}$



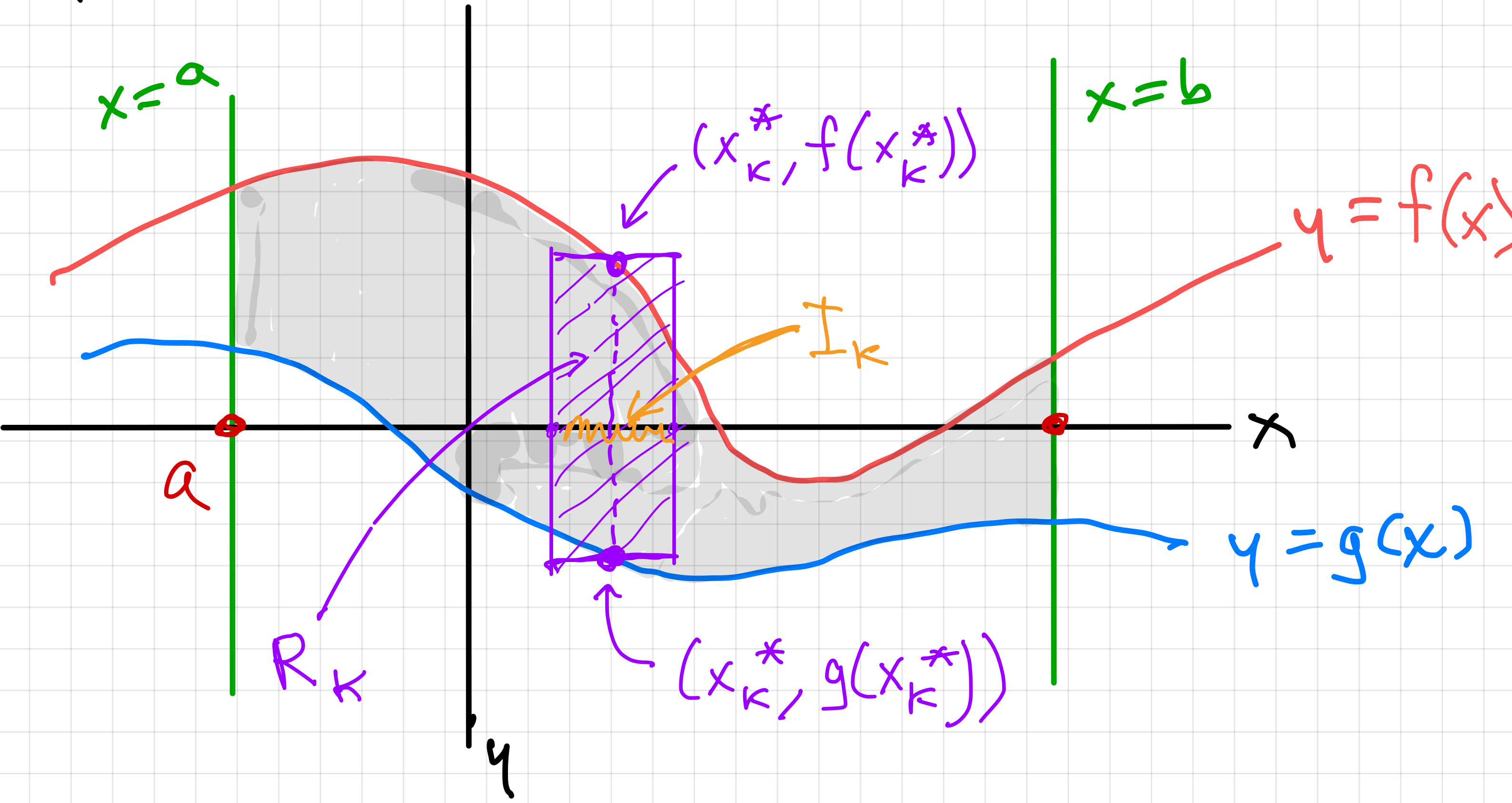
$g(x) \le y \le f(x)$

$\boxed{\text{$g(x) \le f(x)$ means $y=g(x)$ is below $y=f(x)$.}}$

Explaining the area formula for Type I regions

Subdivide the interval $[a,b]$ into N subintervals $I_1, I_2, \dots, I_N$ with length $\Delta x = \frac{b-a}{n}$

In each I_k choose a point x_k^*. Over the interval I_k construct a rectangle R_k with height $f(x_k^*) - g(x_k^*)$ and width Δx as shown:



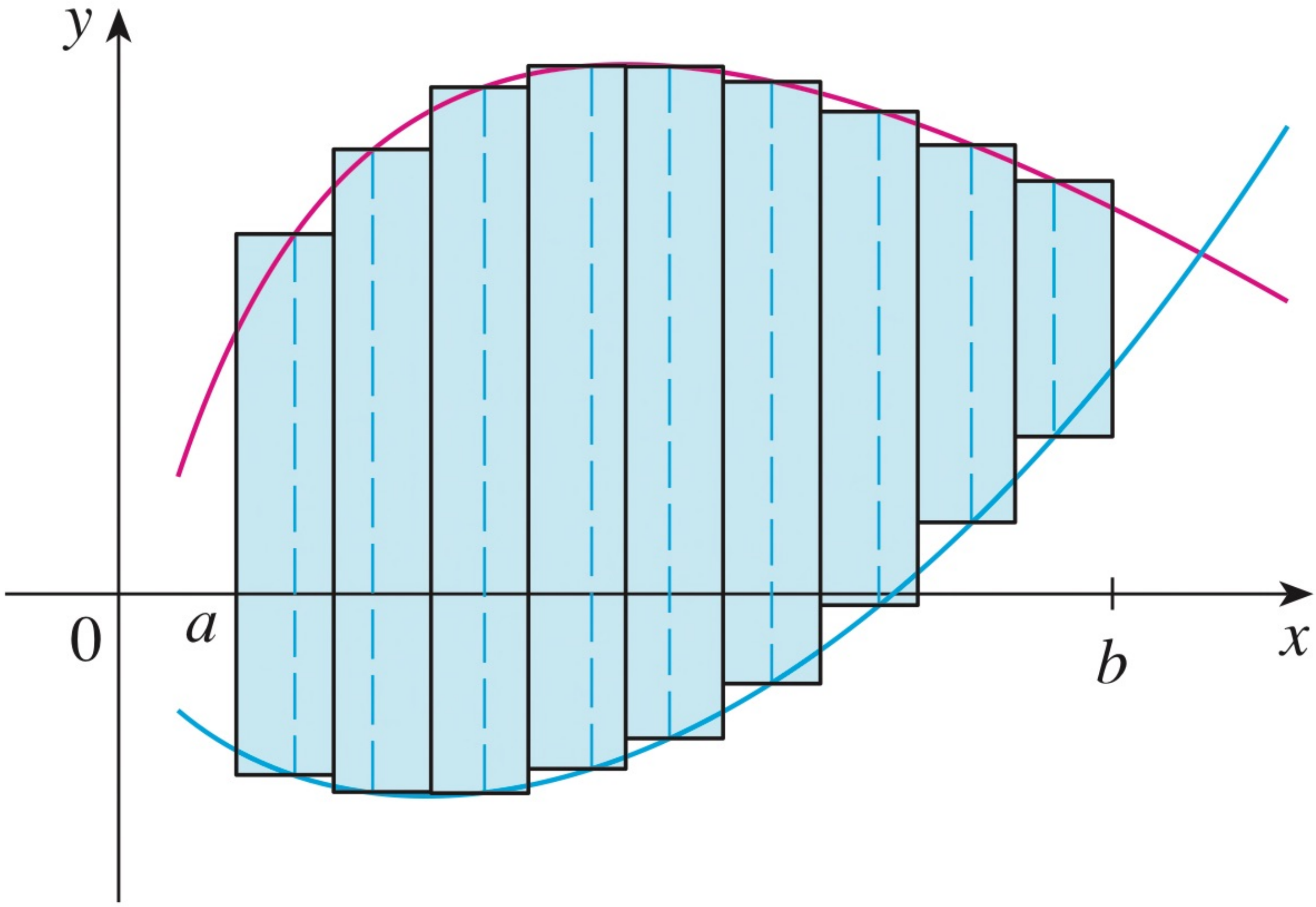
Now Area(S) is approximated by:

$$\sum_{k=1}^{N} \text{area}(R_k) = \sum_{k=1}^{N} \left(f(x_k^*) - g(x_k^*)\right) \Delta x$$

and this is a Riemann Sum for $\int_a^b f(x) - g(x)\, dx$.

Conclude $\text{Area}(S) = \int_a^b f(x) - g(x)\, dx$

Here's a better picture from Stewart:



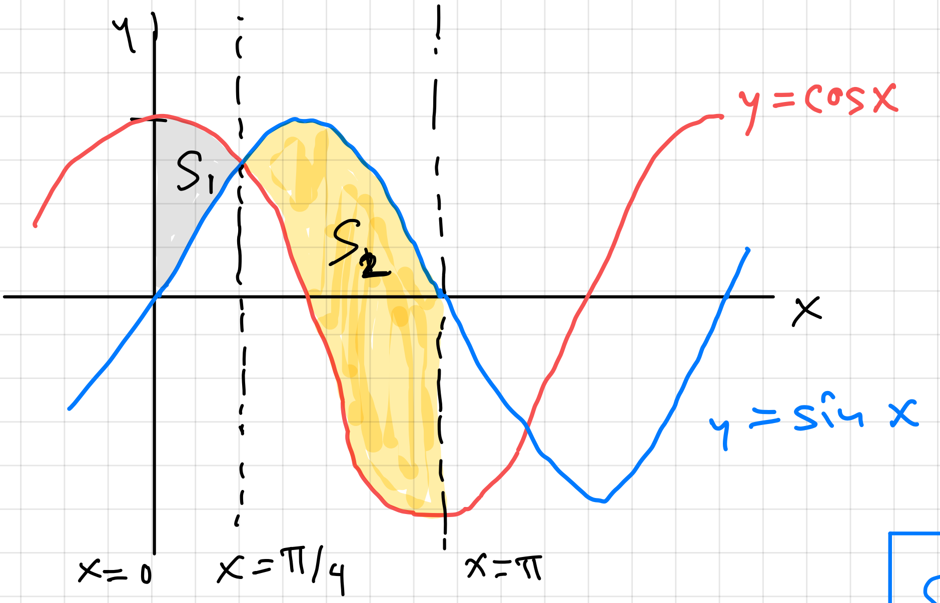
(b) Approximating rectangles

N=9

(Section 5.1 - page 356)

Examples

① Find the area of the region S_1 between $y=\cos(x)$ and $y=\sin(x)$ for $0 \le x \le \pi/4$.



Drawing a sketch can often be helpful.

$$S_2: \begin{cases} \pi/4 \le x \le \pi \\ \cos x \le y \le \sin x \end{cases}$$

$$S_1: \begin{cases} 0 \le x \le \pi/4 \\ \sin x \le y \le \cos x \end{cases}$$

$$\text{Area}(S_1) = \int_0^{\pi/4} \cos x - \sin x \, dx = \sin x + \cos x \Big|_0^{\pi/4} =$$

$$\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - \left(\sin 0 + \cos 0\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1 = \sqrt{2} - 1$$

② Find the area of the region S_2 between $y=\cos(x)$ and $y=\sin(x)$ for $\pi/4 \le x \le \pi$.

$$\text{area}(S_2) = \int_{\pi/4}^{\pi} \sin x - \cos x \, dx = -\cos x - \sin x \Big|_{\pi/4}^{\pi} = \sqrt{2} + 1$$

③ Find the area of the region S_3 between $y=\cos(x)$ and $y=\sin(x)$ for $0 \le x \le \pi$.

$$\text{area}(S_3) = \text{area}(S_1) + \text{area}(S_2) = 2\sqrt{2}$$

watch out for +/- signs!

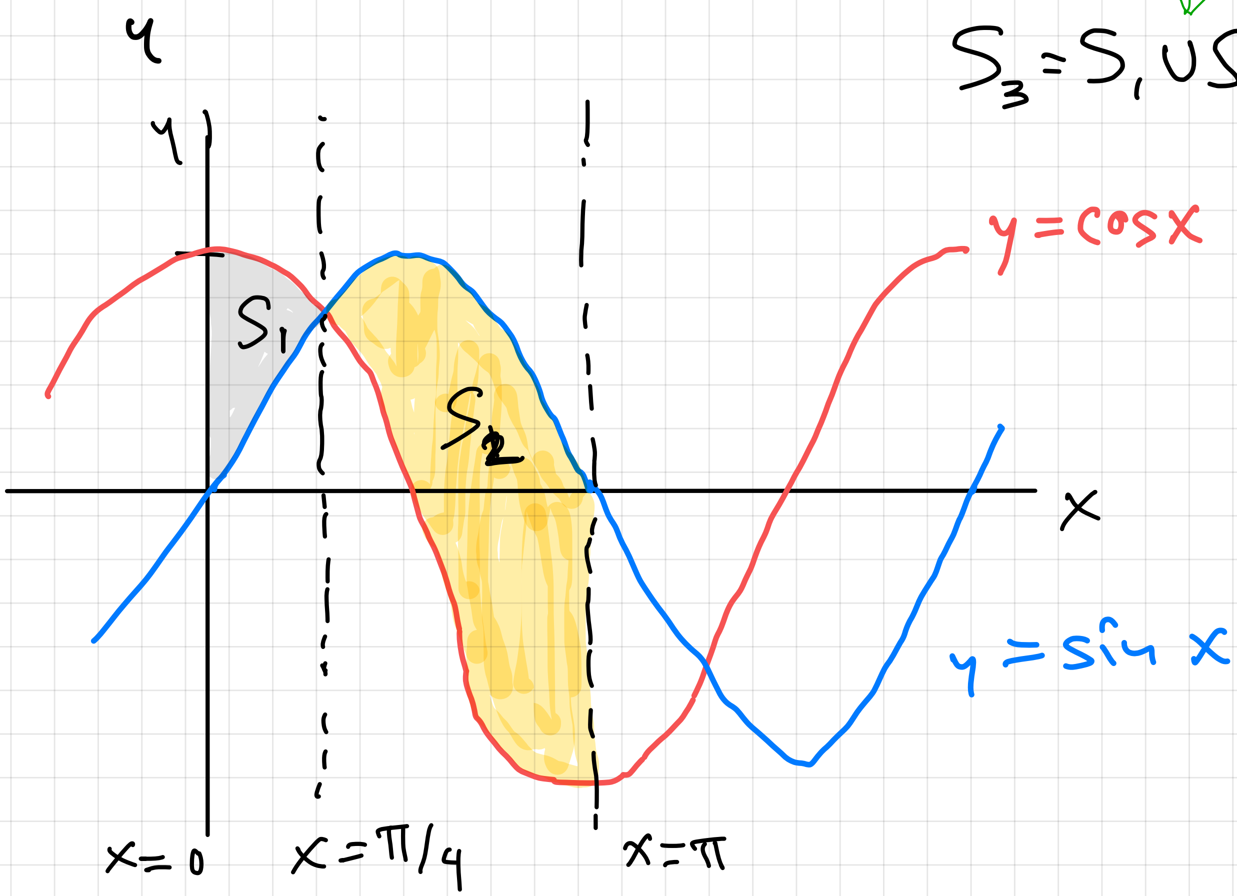
Comments on Example ③.

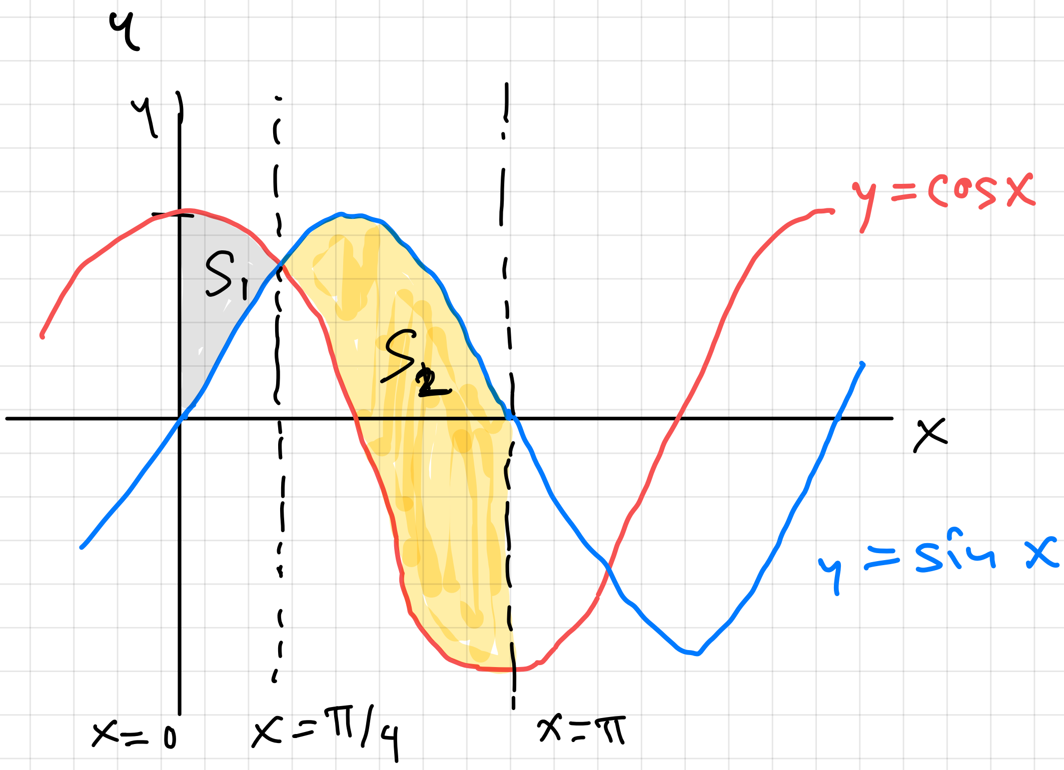
Notice that the region S_3 is <u>not</u> a Type I region because $y=\cos x$ is on top for x between 0 and $\pi/4$, but $y=\cos(x)$ is on top for x between $\pi/4$ and π. However S_3 <u>does</u> decompose into the union of S_1 and S_2, each of which do have type I.

For this reason, $\int_0^\pi \cos x - \sin x \, dx = -2$ does not equal the area of S_3.

union symbol

$$S_3 = S_1 \cup S_2$$





However $\int_0^{\pi} |\cos x - \sin x|\, dx$ does represent the area of S_3, as discussed on page 360 of Stewart's book. But this observation entails the exact same calculation as before, because:

For x between 0 and π, we have

$$|\cos x - \sin x| = \begin{cases} \cos x - \sin x & \text{if } 0 \le x \le \pi/4 \\ \sin x - \cos x & \text{if } \pi/4 < x \le \pi \end{cases}$$

So,

$$\int_0^{\pi} |\cos x - \sin x|\, dx = \int_0^{\pi/4} \cos x - \sin x \, dx + \int_{\pi/4}^{\pi} \sin x - \cos x \, dx$$

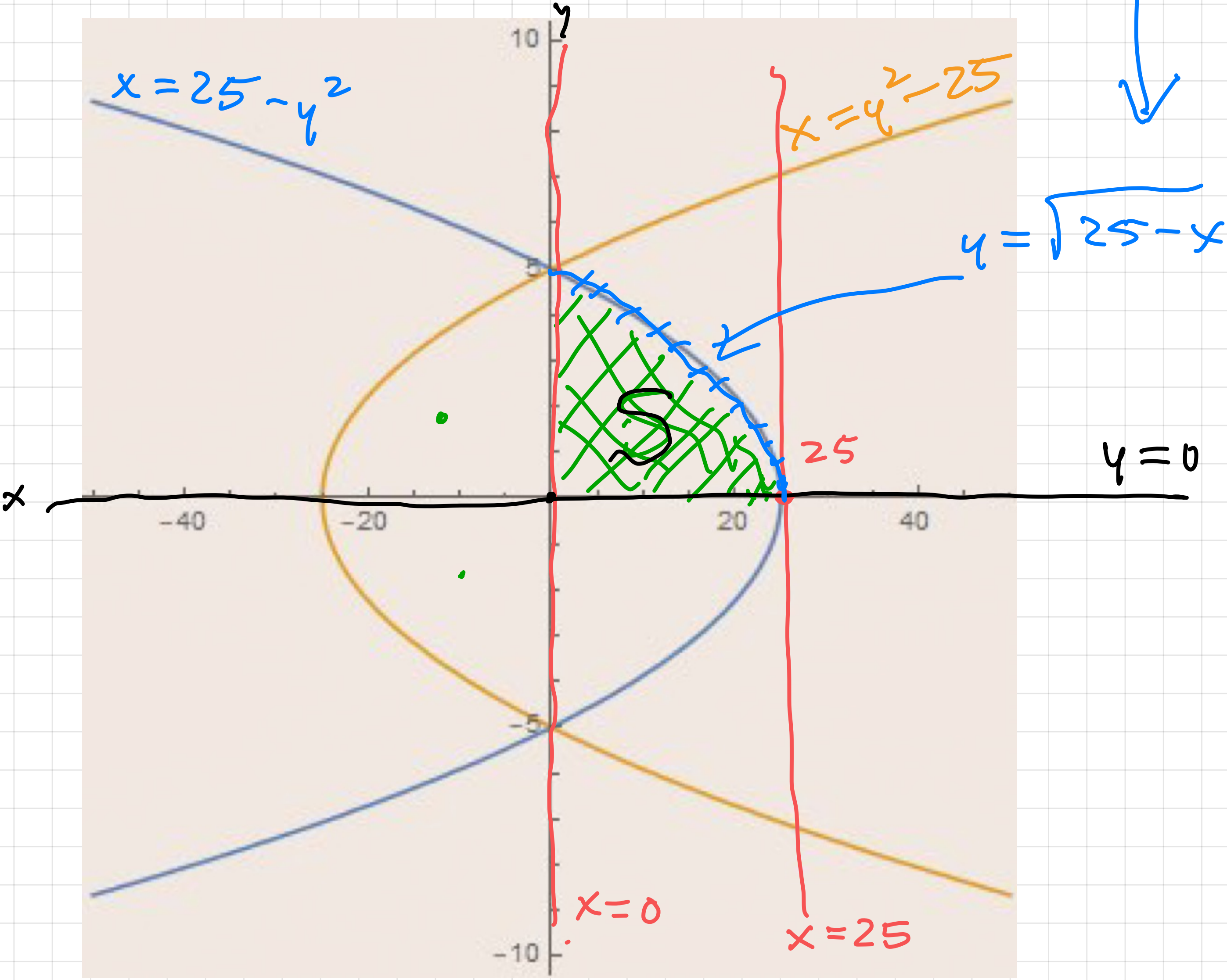
$$= \text{Area}(S_1) + \text{Area}(S_2)$$

Example: Find the area of the region between $x = 25 - y^2$ and $x = y^2 - 25$.

$y^2 = 25 - x$

$y = \sqrt{25-x}$

these are two parabolas:



$$S: \begin{cases} 0 \le x \le 25 \\ 0 \le y \le \sqrt{25-x} \end{cases}$$

use a substitution

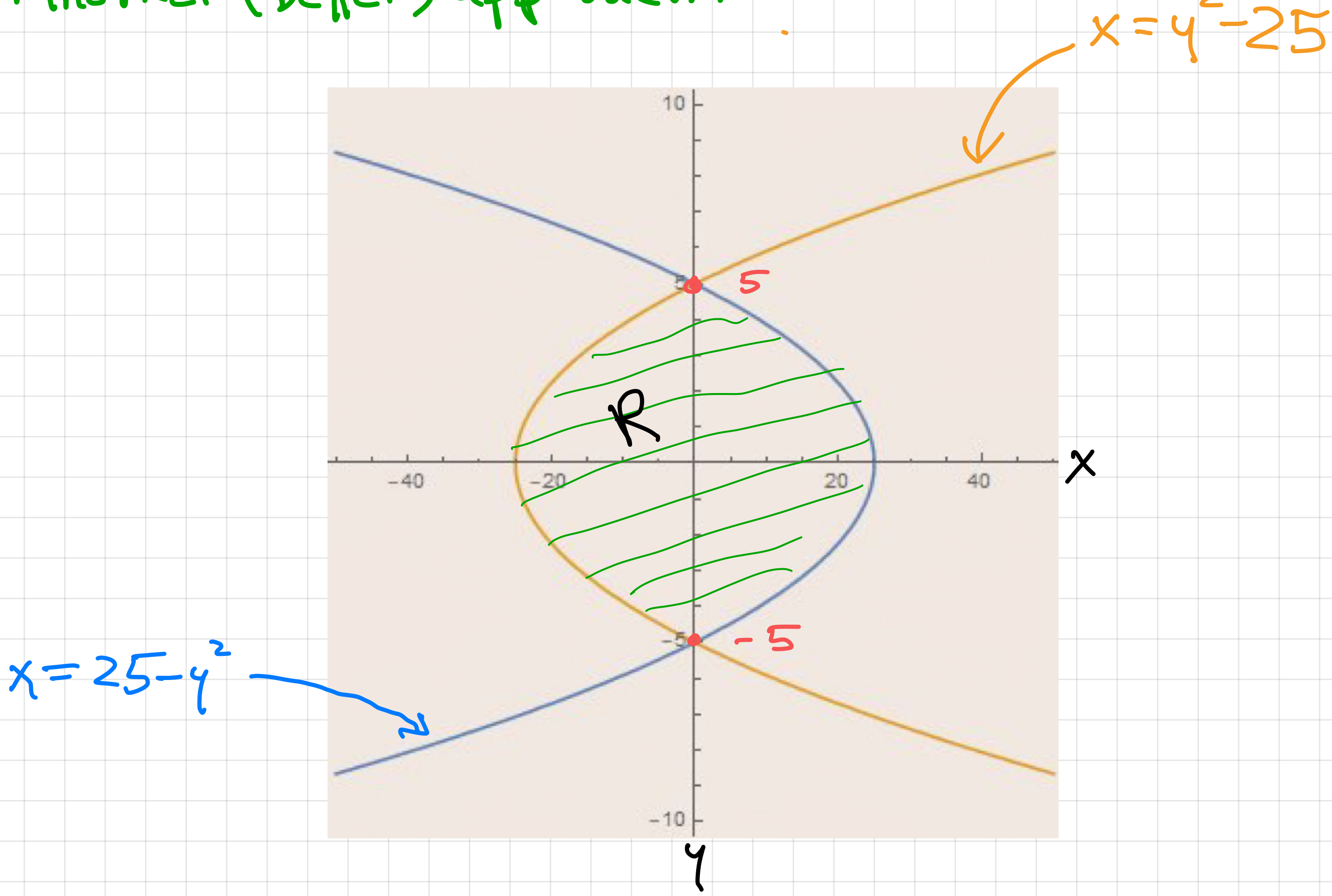
$$\frac{250}{3}$$

$$\text{area}(S) = \int_0^{25} \sqrt{25-x} - 0 \, dx$$

Final answer $\frac{1000}{3}$ is four times the area of S by symmetry considerations.

Example: Find the area of the region R between $x = 25 - y^2$ and $x = y^2 - 25$.

Another (better) approach:



The region R can be described by inequalities:

$$R : \begin{cases} -5 \le y \le 5 \\ y^2 - 25 \le x \le 25 - y^2 \end{cases}$$

Think of this as a type I region where the roles of x and y have been switched. Then

$$\text{Area}(R) = \int_{-5}^{5} (25-y^2) - (y^2-25)\, dy = \int_{-5}^{5} 50 - 2y^2 \, dy = 50y - \tfrac{2}{3}y^3 \Big|_{-5}^{5}$$

$$= \left(250 - \tfrac{250}{3}\right) - \left(-250 + \tfrac{250}{3}\right) = 500 - \tfrac{500}{3} = 1000/3$$