Integration Recap

$$\int_a^b f(x)\, dx$$

integrand

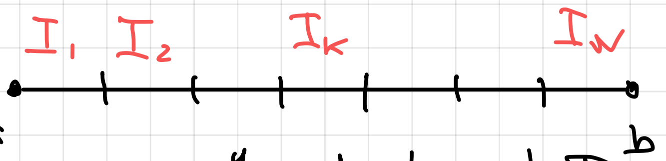
integral symbol

indicates the variable of integration.

limits of integration

$$\boxed{\text{Definition: } \int_a^b f(x)\,dx = \lim_{N\to\infty} R_N}$$

$$R_N = \sum_{k=1}^{N} f(x_k^*)\,\Delta x$$

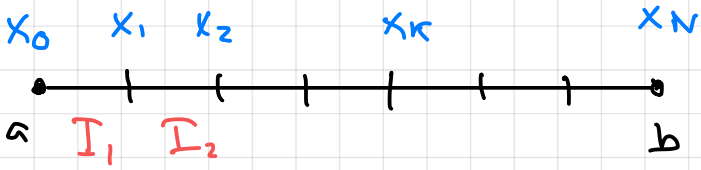


where $\Delta x = (b-a)/N$ and x_k^* is in k^{th} subinterval I_k

Write $x_k = a + k\,\Delta x$

$x_0 = a + 0\,\Delta x$
$= a$

$x_N = a + N\Delta x$
$= b$



Then $I_k = [x_{k-1}, x_k]$ closed interval

left Riemann Sum: Choose $x_k^* = x_{k-1}$ = left endpt.

right Riemann Sum: Choose $x_k^* = x_k$

midpoint Riemann Sum: Choose $x_k^* = (x_{k-1} + x_k)/2$

In certain special cases $\int_a^b f(x)\,dx$ can be interpreted as the area of a certain region in the xy-plane.

Basic Properties of Integrals

Very important

see Stewart p. 313 – 315

① $\int_a^b c\,dx = c(b-a)$ if c is constant

② $\int_a^b cf(x) + d\,g(x)\,dx = c\int_a^b f(x)\,dx + d\int_a^b g(x)\,dx$

if c, d are constants

③ $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

④ $\int_a^a f(x)\,dx = 0$

lower bound for $f(x)$ on $[a,b]$

⑤ If $m \le f(x) \le M$ when $a \le x \le b$ then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

⑥ $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

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⑥ $\int_b^a f(x)\,dx + \int_a^b f(x)\,dx \overset{③}{=} \int_b^b f(x)\,dx \overset{④}{=} 0$

$$\Longrightarrow \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

Shows property ⑥ follows from ③ and ④.

Property 3

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

Can also write this as:

$$\int_c^b f(x)\,dx = \int_a^b f(x)\,dx - \int_c^b f(x)\,dx$$

The Fundamental Theorem of Calculus (FTC)

How can we calculate $\int_a^b f(x)\,dx$?

Let's study $F(t) = \int_a^t f(x)\,dx$.

What is $F'(t)$?

$$\frac{F(t+h)-F(t)}{h} = \frac{1}{h}\left(\int_a^{t+h} f(x)\,dx - \int_a^t f(x)\,dx \right)$$

$$= \frac{1}{h}\int_t^{t+h} f(x)\,dx \approx \frac{1}{h}\int_t^{t+h} f(t)\,dx =$$

If h is very small then the minimum and max. of $f(x)$ on $[t, t+h]$ will very close to $f(t)$.

$$= \frac{1}{h} f(t)\left((t+h)-t\right) = \frac{1}{h} f(t)\,h = f(t)$$

Conclude This shows that

$F'(t) = f(t) =$ integrand of original integral !!

FTC - Version 1: If $F(t) = \int_a^t f(x)\,dx$

then (i) $F'(t) = f(t)$, and

(ii) $F(a) = 0$

example $\int_{-2}^{1} x^3\,dx = ??$

Let $F(t) = \int_{-2}^{t} x^3\,dx$ then FTC says

(i) $F'(t) = t^3$, and (ii) $F(-2) = 0$.

By (i) it must be that:

$F(t) = \frac{1}{4}t^4 + C$ for some constant C.

By (ii) it must be that $C = -4$.

(check: $\frac{1}{4}(-2)^4 - 4 = \frac{1}{4}(16) - 4 = 0$)

So $\boxed{F(t) = \frac{1}{4}t^4 - 4}$

Now observe that

$$\int_{-2}^{1} x^3\,dx = F(1) = \frac{1}{4}(1)^4 - 4 = -\frac{15}{4}$$

It's a magic trick!

The previous example suggests that many integrals can be calculated using "antiderivatives", but the process is a bit circuitous. A more direct approach uses:

<u>FTC - Version 2</u> If $F(x)$ is any antiderivative for $f(x)$ then

convenient shorthand notation

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_{x=a}^{b}$$

<u>example</u> To calculate $\int_{-2}^{1} x^3\,dx$ we can see that $F(x) = \frac{1}{4}x^4$ is an antiderivative of $f(x) = x^3$ so

$$\int_{-2}^{1} x^3\,dx = \frac{1}{4}x^4\Big|_{x=-2}^{1} = \frac{1}{4} - \frac{1}{4}(-2)^4 = -\frac{15}{4}$$

To clarify:

<u>Definition</u>: $F(x)$ is an <u>antiderivative</u> for $f(x)$ provided that $F'(x) = f(x)$.

Problem (using FTC)

Find a formula for $F(x)$ where

ⓐ $F(x) = \int_1^x -7t^2 + 3t - 2 \, dt$

ⓑ $F(x) = \int_0^x -7t^2 + 3t - 2 \, dt$

ⓐ By FTC, $F'(x) = f(x) = -7x^2 + 3x - 2$

So $F(x)$ is an antiderivative of $f(x)$

$$f(x) = -7x^2 + 3x - 2$$

$$\frac{d}{dx}\left(\underbrace{-\frac{7}{3}x^3 + \frac{3x^2}{2} - 2x}\right) = \overset{f(x)}{\overset{\parallel}{-7x^2 + 3x - 2}}$$

Take $F(x) = -\frac{7}{3}x^3 + \frac{3x^2}{2} - 2x + C$

then $F(1) = -\frac{7}{3} + \frac{3}{2} - 2 + C = -\frac{17}{6} + C$

Take $C = 17/6$

$$F(x) = -\frac{7}{3}x^3 + \frac{3x^2}{2} - 2 + \frac{17}{6}$$

ⓑ Take $F(x) = -\frac{7}{3}x^3 + 3\frac{x^2}{2} - 2x$. Then $F'(x) = -7x^2 + 3x - 2$ and $F(0) = 0$.