True or False?

① An equation $y = f(x)$ can always be solved for x giving $x = g(y)$ where $g(y)$ is a function. False!

② Every function $f(x)$ has an inverse function. False!

③ If $f(x)$ has an inverse function $f^{-1}(x)$ then $\text{domain}(f^{-1}) = \text{range}(f)$ and $\text{range}(f^{-1}) = \text{Domain}(f)$. True

④ If the graph $y = f(x)$ satisfies HLP then f has an inverse function. True

⑤ If $f(x)$ has an inverse function then $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. True

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The "identity function is $I(x) = x$. So property ⑤ can be written as

$$f \circ f^{-1} = I = f^{-1} \circ f$$

Definition of natural logarithm function

$$\boxed{\text{For } x>0, \quad \ln(x) = \int_1^x \frac{1}{t}\,dt}$$

note: Many authors write $\log(x)$ instead of $\ln(x)$. In WebWork you may see either $\ln(x)$ or $\log(x)$.

The FTC: $\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$ shows that:

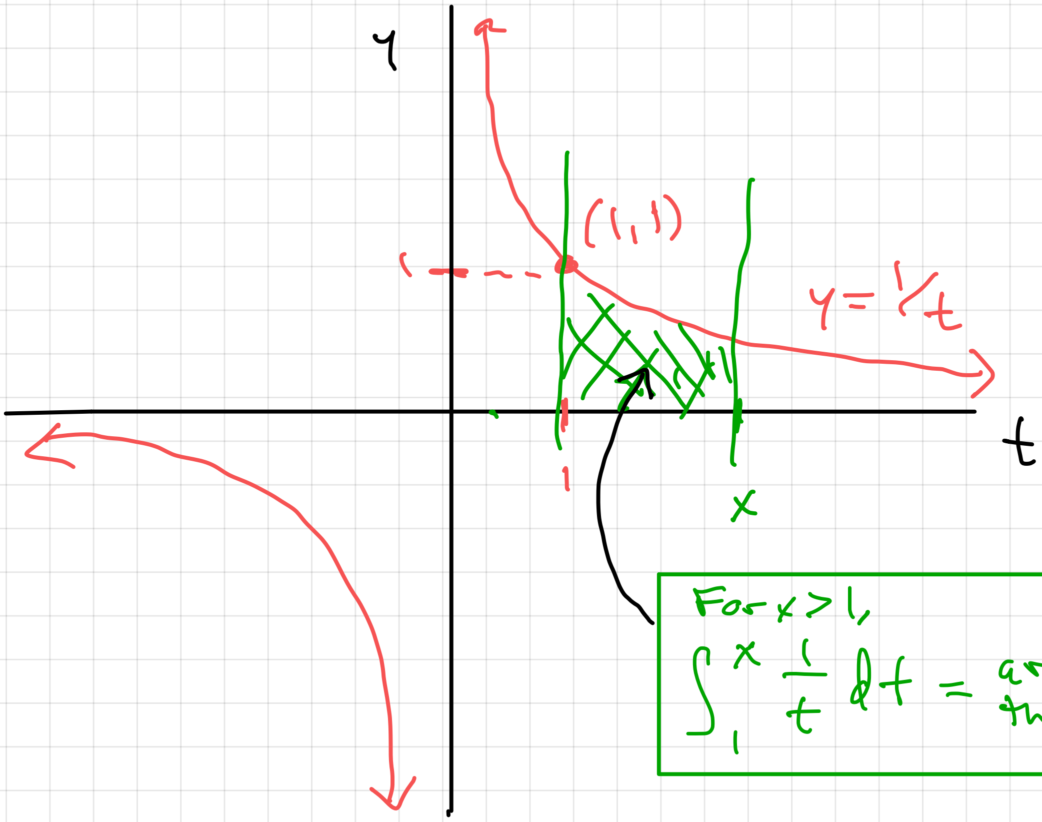
$$\boxed{\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}}$$

and

$$\boxed{\int \frac{1}{x}\,dx = \ln(x) + C, \quad x>0}$$

First Observations:

- $\ln(x)$ is only defined when $x>0$ because $\frac{1}{t} = \text{DNE}$ when $t=0$.
  So domain$(\ln) = (0,\infty)$
- range$(\ln) = \mathbb{R} = (-\infty,\infty)$
- $\ln(1) = 0$ because $\int_1^1 \frac{1}{t}\,dt = 0$.
- When $x>1$, $\ln(x)>0$.
- When $0<x<1$, $\ln(x)<0$.



For $x>1$,
$$\int_1^x \frac{1}{t}\,dt = \text{area of this region} > 0$$

For $0<x<1$, $\int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt < 0$

Consider the function $f(x) = \ln(x^2)$, $x>0$

By the chain rule
$$f'(x) = \frac{1}{x^2}\frac{d}{dx}\left[x^2\right] = \frac{2x}{x^2} = \frac{2}{x}$$

But it's also true that
$$\frac{d}{dx}\left[2\ln(x)\right] = \frac{2}{x}$$

We conclude from this that
$$\ln(x^2) = 2\ln(x) + C$$
and taking $x=1$ gives $C = 2\ln(1) + C = \ln(1^2) = 0$

This shows that
$$\boxed{\ln(x^2) = 2\ln(x)\,,\quad x>0}$$

In fact we'll see that $\ln(x)$ has many interesting algebraic properties.

Inverse Functions

Only "some", not "all"!

Some functions $f(x)$ are linked with an "inverse function" $f^{-1}(x)$, and often the inverse function has interesting properties.

On pages 400–401, Stewart writes:

2 Definition Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

and

"one-to-one" ≡ "1-1"

1 Definition A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

Examples (described on coming pages)

① $f(x) = -12x + 1$

② $f(x) = x + \cos(x)$

These are 1-1 and have inverse functions.

③ $f(x) = x^2$

④ $f(x) = x^3 - 27x + 10$

These are not 1-1 and do not have inverse functions.

① $f(x) = -12x + 1$

Try to solve $y = f(x)$ for x.

$$y = -12x + 1$$

subtract 1

$$-12x = y - 1$$

divide by -12

$$x = -\frac{1}{12}y + \frac{1}{12}$$

So $f(x)$ is one-to-one and $f^{-1}(y) = -\frac{1}{12}y + \frac{1}{12}$

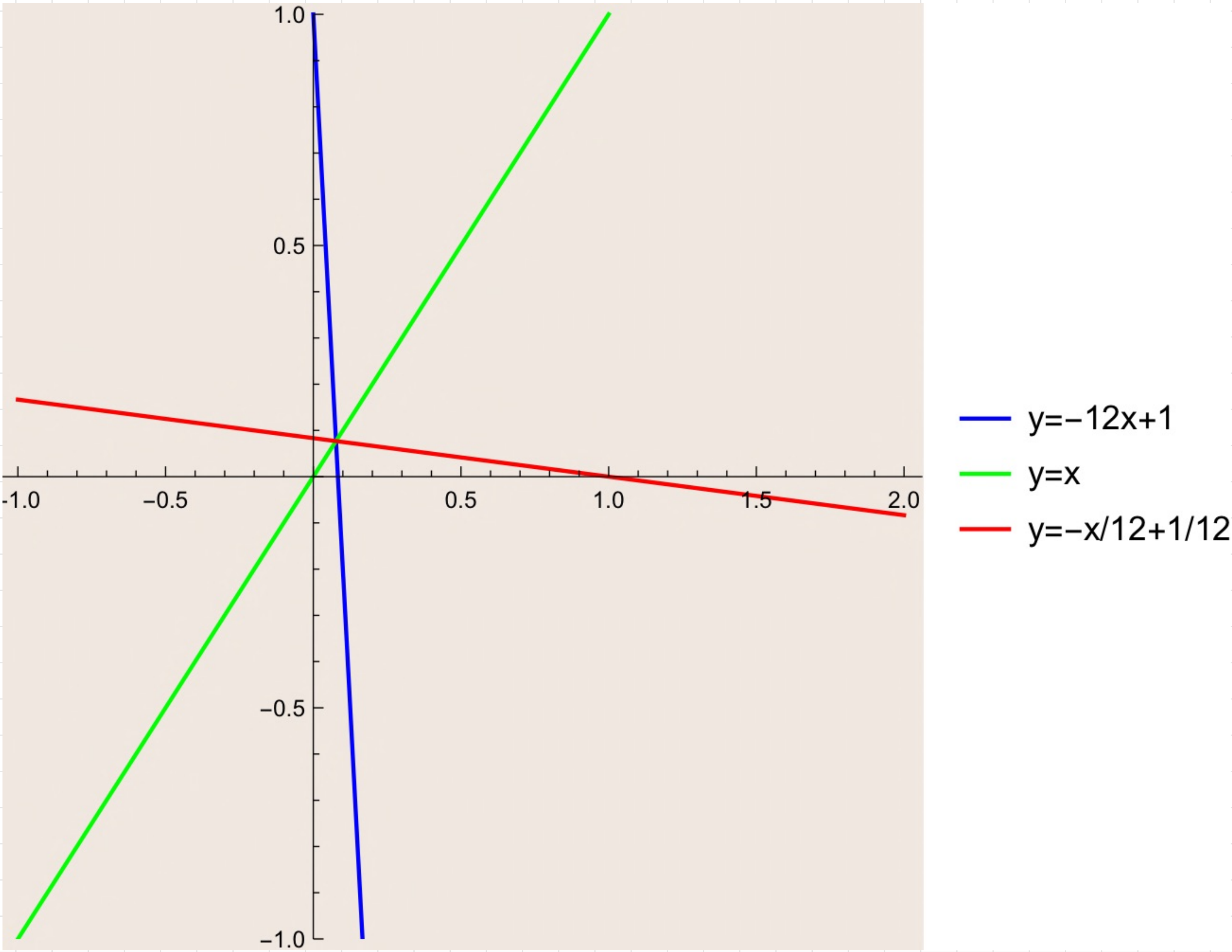
Or $$f^{-1}(x) = -\frac{1}{12}x + \frac{1}{12}$$

What does the graph of $y = f^{-1}(x)$ look like?

Answer: The graph of $y = f^{-1}(x)$ is obtained from $y = f(x)$ by taking its mirror image in the line $y = x$.

Why? Because when we rewrote the equation $y = f(x)$ as $x = -\frac{1}{12}y + \frac{1}{12}$ in ⊛ we didn't change the solution set. (An ordered pair (x, y) is a solution to $y = f(x)$ exactly when its a solution to $x = -\frac{1}{12}y + \frac{1}{12}$.) But interchanging x and y to get the equation $y = -\frac{1}{12}x + \frac{1}{12}$ has the effect of taking the mirror image across $y = x$.

Graphs of $f(x) = -12x+1$ and $f^{-1}(x) = -\frac{1}{12}x + \frac{1}{12}$:



- The blue and red graphs are mirror images across the (green) line $y = x$.
- This graph has 'aspect ratio' $= \dfrac{\text{length } y\text{-unit}}{\text{length } x\text{-unit}} \approx 1.5$

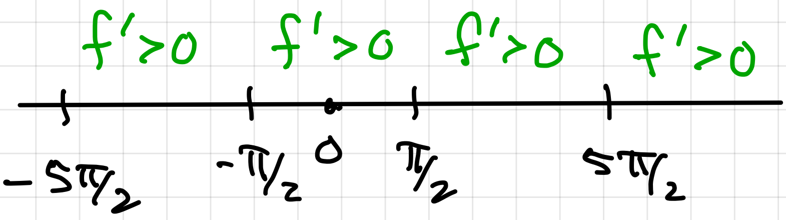
because it makes the picture easier to read.
(This means the line $y = x$ forms an angle larger than $45°$ with the x-axis.)

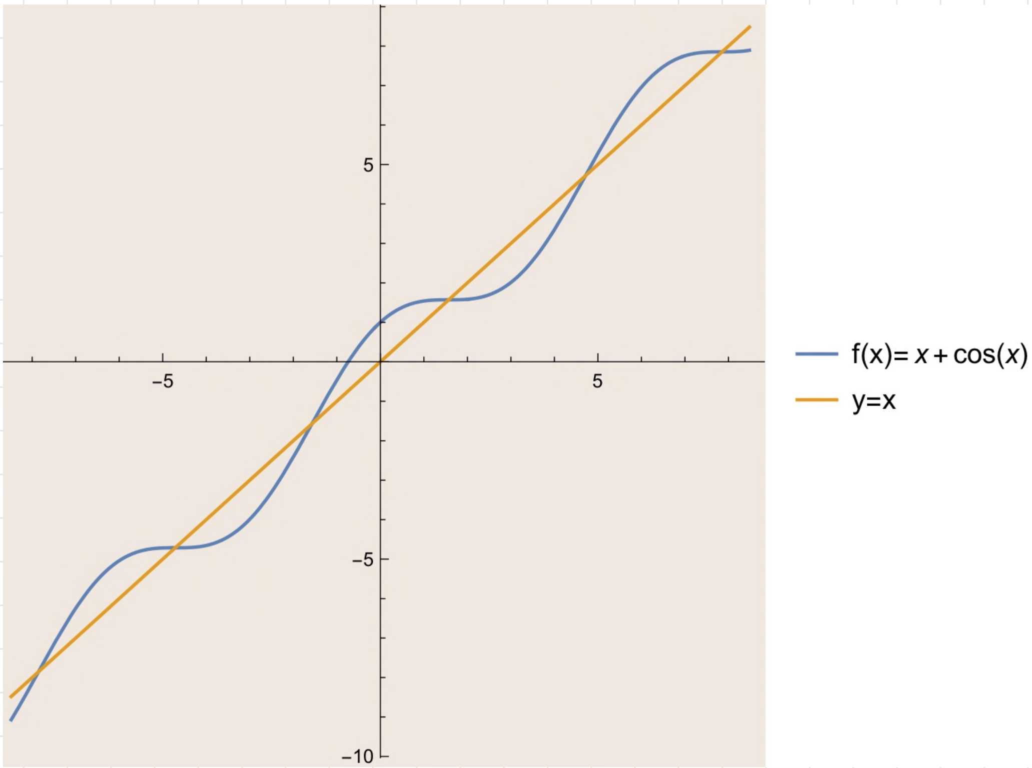
How to determine when $f(x)$ is one-to-one.

(next class ...)

② $f(x) = x + \cos(x)$

$f'(x) = 1 - \sin(x)$





Critical #'s

$1 - \sin x = 0$

$\sin x = 1$

$x = \ldots -\pi/2, \pi/2, \frac{5\pi}{2}, \ldots$

So

$f(x)$ is a strictly increasing function

$\Rightarrow f(x)$ is 1-1

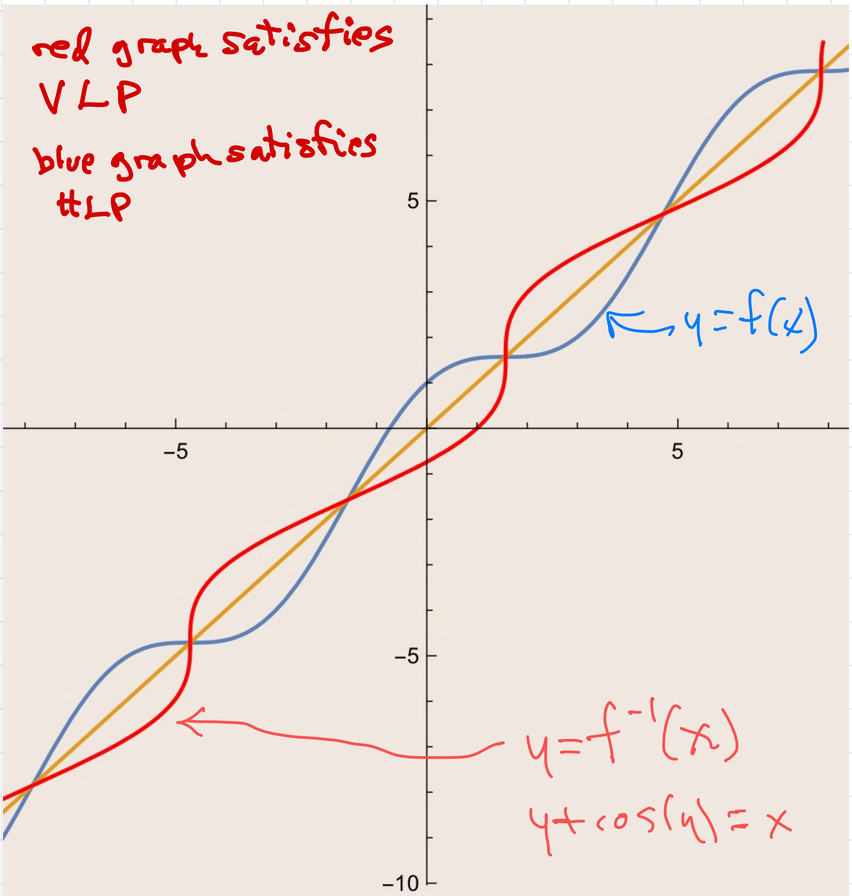
$\Rightarrow f^{-1}(x)$ exists

But solve

$y = x + \cos(x)$

for x?

can't do it!



Conclude: $f^{-1}(x)$ exists but there is no nice formula for it.

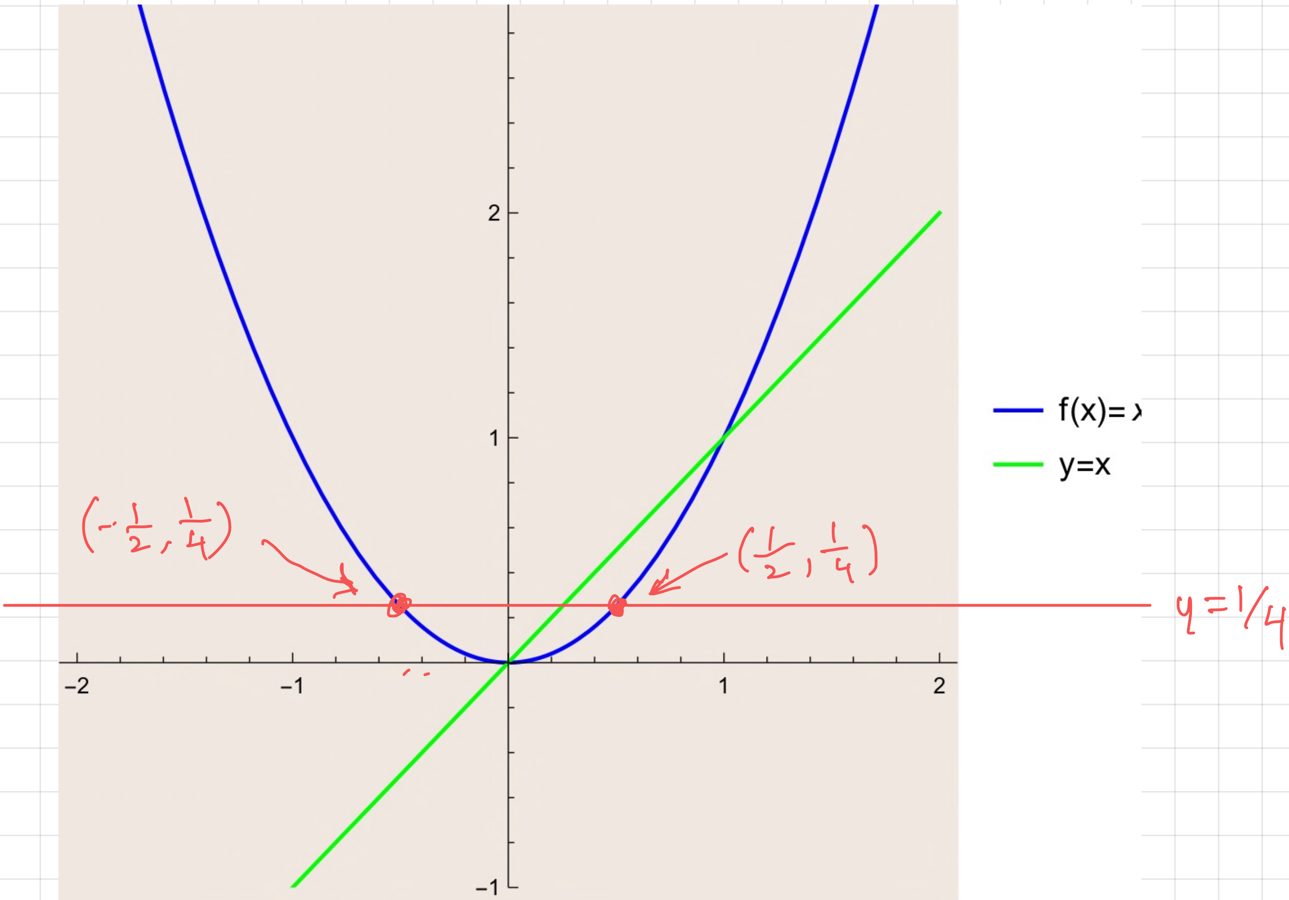
However the graph of $y = f^{-1}(x)$ is the graph of

$y + \cos(y) = x$.

③ $f(x) = x^2$

<u>example</u> $f(x) = x^2$ is <u>not</u> one-to-one because $f(\frac{1}{2}) = \frac{1}{4}$ and $f(-\frac{1}{2}) = \frac{1}{4}$ (but $\frac{1}{2} \neq -\frac{1}{2}$).

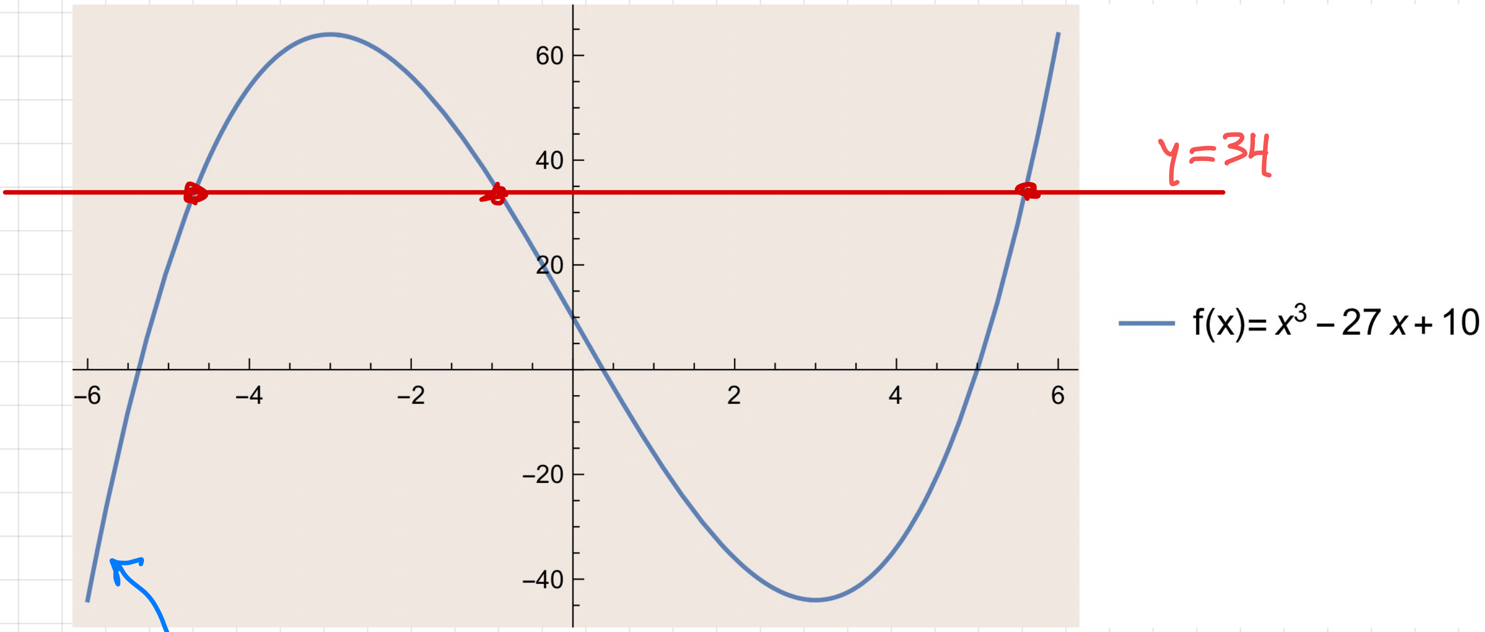
(more on this next class)



(4) $f(x) = x^3 - 27x + 10$

Use calculus to sketch the graph $y = f(x)$ . . . .

(next class . . .)



Graph of $f(x)$ doesn't satisfy HLP, so there is no inverse function.

You could also observe, for example, that

$$f(0) = f(3\sqrt{3}) = f(-3\sqrt{3}) = 10$$

but $0 \neq 3\sqrt{3} \neq -3\sqrt{3}$