1 Definition A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

abbreviate:

$$1\text{-}1 \equiv \text{one-to-one}$$

One-to-one functions are the functions which have inverse functions.

How can we tell if $f(x)$ is one-to-one?

- If the equation $y=f(x)$ can be solved uniquely for x then $f(x)$ is one-to-one.
- If there are numbers x_1 and x_2 with $f(x_1)=f(x_2)$ and $x_1 \neq x_2$ then $f(x)$ is not one-to-one.
- If it can be shown algebraically that $f(x_1)=f(x_2) \Rightarrow x_1=x_2$ then $f(x)$ is one-to-one.
- If $f(x)$ is a strictly increasing or strictly decreasing function on its domain then $f(x)$ is one-to-one.
- If the graph of $y=f(x)$ satisfies HLP then $f(x)$ is one-to-one.
- If the graph of $y=f(x)$ does not satisfy HLP then $f(x)$ is not a one-to-one function.
- If there is a function $g(x)$ with the property that $f(g(x))=x$ and $g(f(x))=x$ then $f(x)$ is one-to-one.

$\boxed{\text{One-to-one functions are the functions which have inverse functions.}}$

Examples (described in coming pages)

① $f(x) = -12x + 1$

② $f(x) = x + \cos(x)$

These are 1-1 and have inverse functions.

③ $f(x) = x^2$

④ $f(x) = x^3 - 27x + 10$

These are not 1-1 and do not have inverse functions.

①

$y = -12x + 1$

solve for x

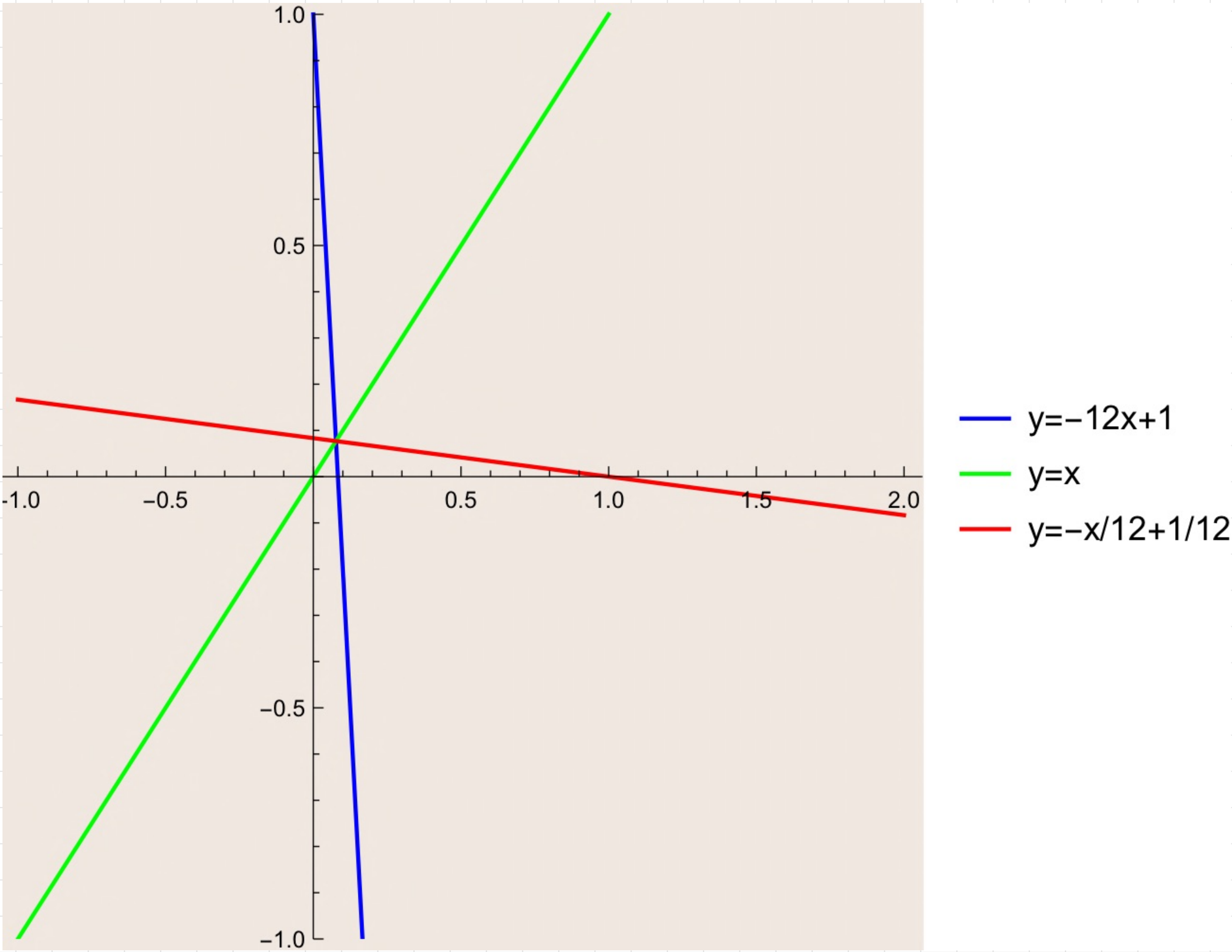
$x = -\frac{1}{12}y + \frac{1}{12}$

success means that $f(x) = -12x + 1$ is one-to-one, and also shows that

$$f^{-1}(x) = -\frac{1}{12}x + \frac{1}{12}$$

① pictorially:

Graphs of $f(x) = -12x + 1$ and $f^{-1}(x) = -\frac{1}{12}x + \frac{1}{12}$:

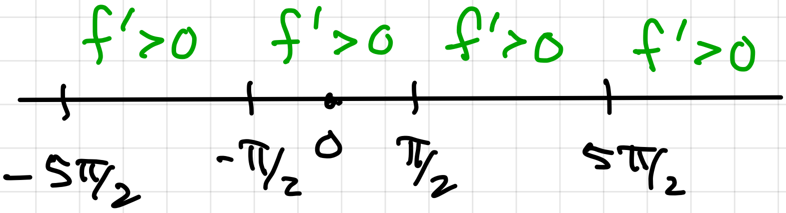


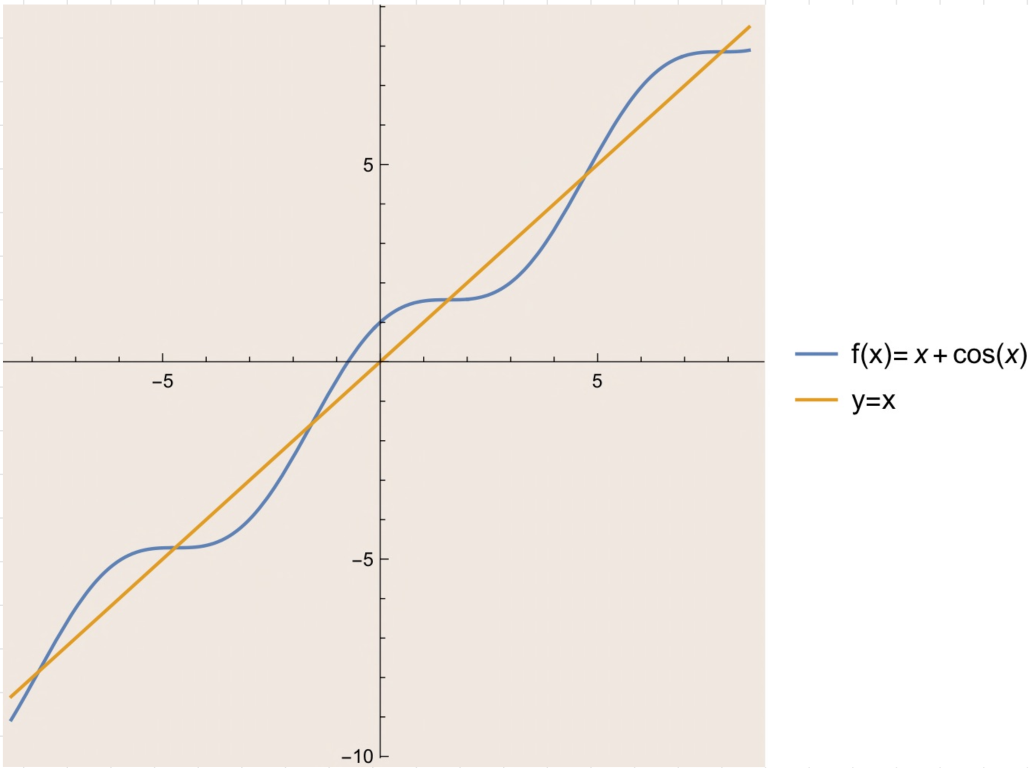
- The blue and red graphs are mirror images across the (green) line $y = x$.
- This graph has 'aspect ratio' $= \dfrac{\text{length } y\text{-unit}}{\text{length } x\text{-unit}} \approx 1.5$

because it makes the picture easier to read.
(This means the line $y=x$ forms an angle larger than $45°$ with the x-axis.)

② $f(x) = x + \cos(x)$

$f'(x) = 1 - \sin(x)$





Critical #'s

$1 - \sin x = 0$

$\sin x = 1$

$x = \ldots -\pi/2, \pi/2, \frac{5\pi}{2}, \ldots$

So

$f(x)$ is a strictly increasing function

$\Rightarrow f(x)$ is 1-1

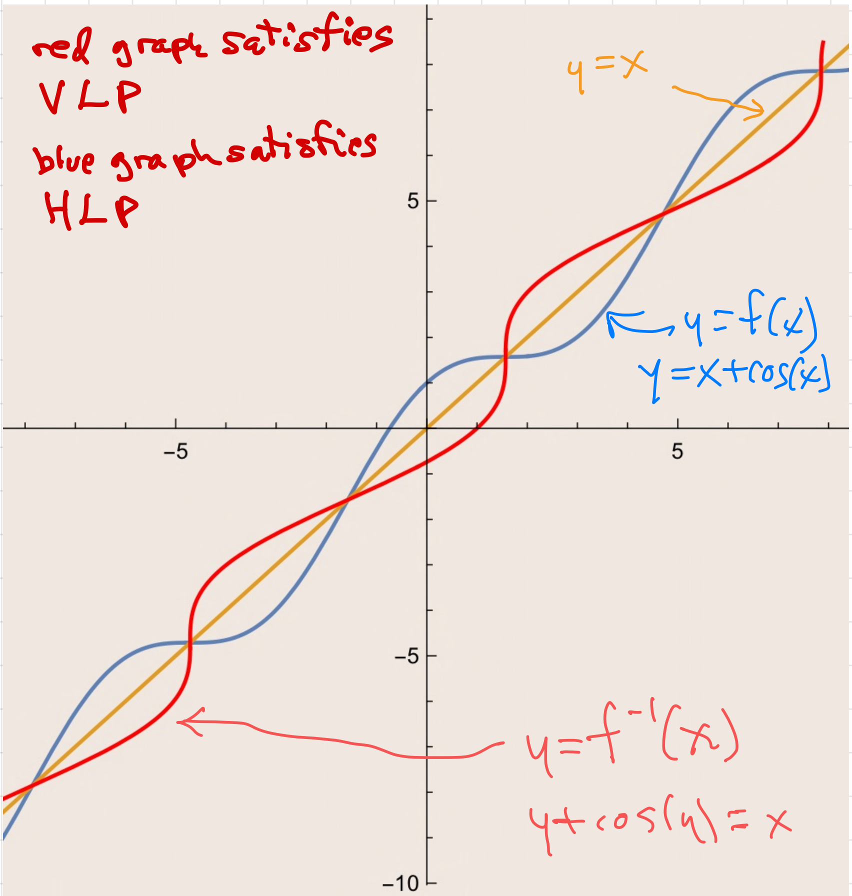
$\Rightarrow f^{-1}(x)$ exists

But solve

$y = x + \cos(x)$

for x?

can't do it!



Conclude: $f^{-1}(x)$ exists but there is no nice formula for it.
However the graph of $y = f^{-1}(x)$ is the graph of
$y + \cos(y) = x$.

③ $f(x) = x^2$

<u>example</u> $f(x) = x^2$ is <u>not</u> one-to-one because $f(\frac{1}{2}) = \frac{1}{4}$ and $f(-\frac{1}{2}) = \frac{1}{4}$ (but $\frac{1}{2} \neq -\frac{1}{2}$).

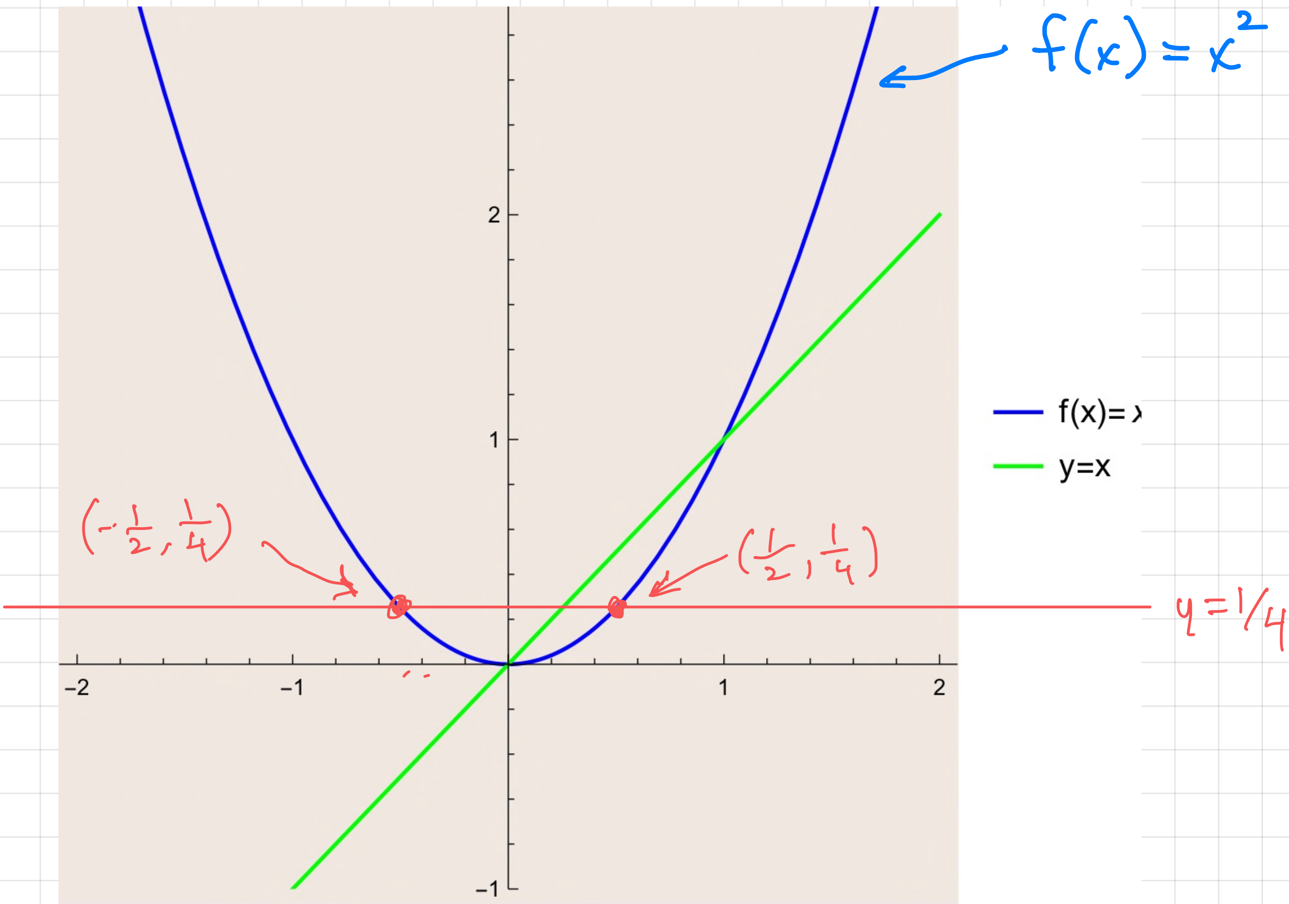
So $f(x)$ is not one-to-one

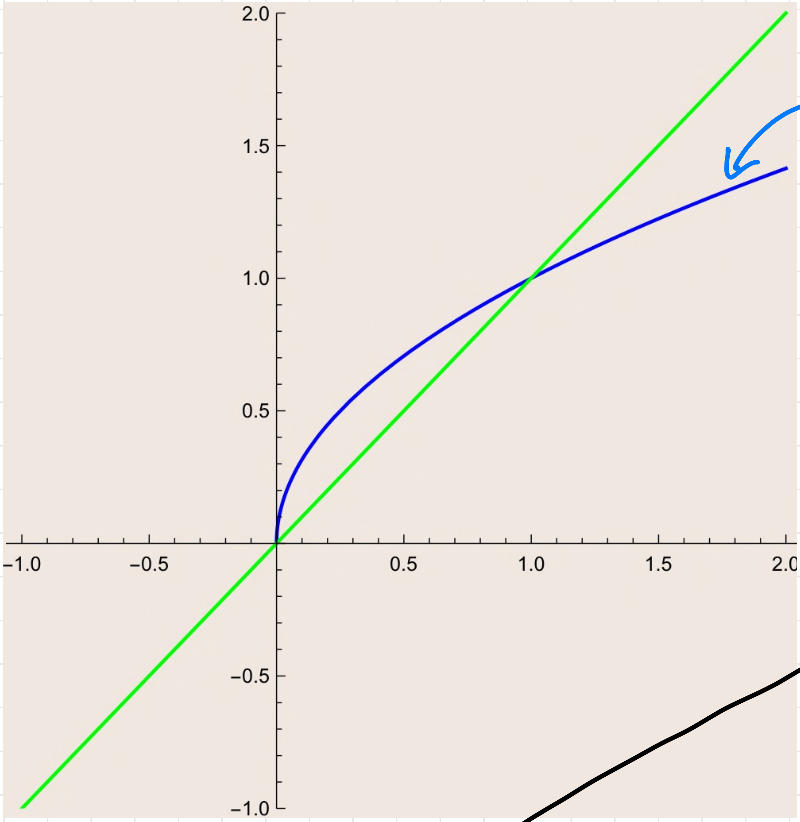
Graph of $y = x^2$ does not satisfy HLP.

$y = x^2$ ← solve for x

$x = \pm\sqrt{y}$ ← not a <u>unique</u> value for y!

not a function of y!





related example

$f(x) = \sqrt{x}$

This graph satisfies HLP, so $\sqrt{x}$ has an inverse function.

and

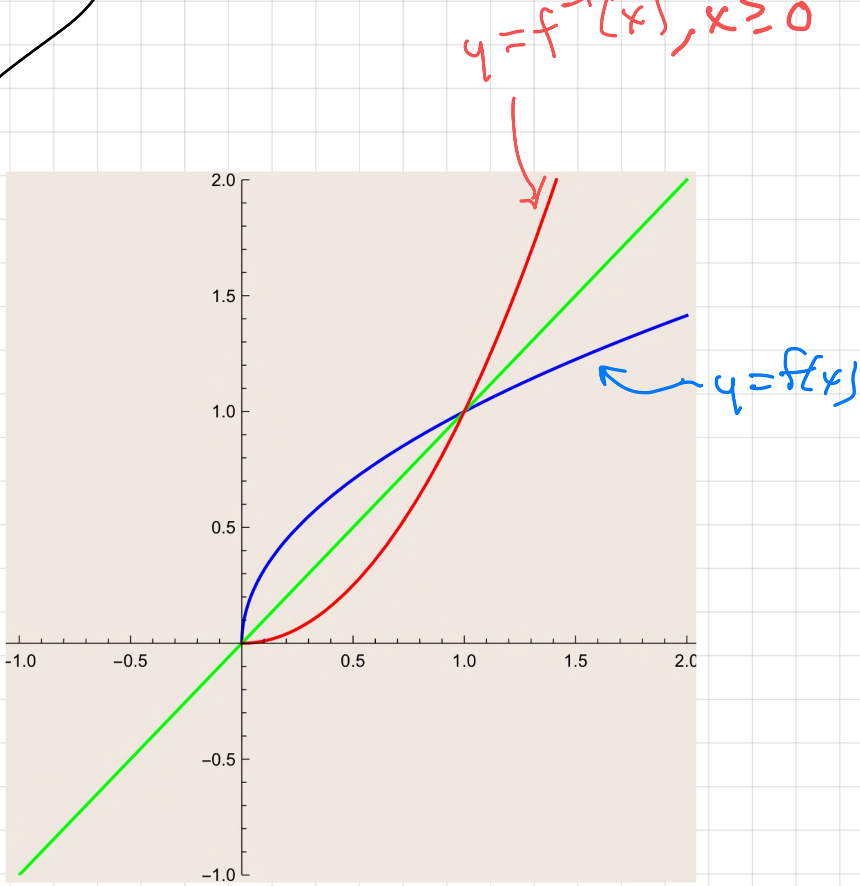
$f^{-1}(x) = x^2, \ x \geq 0$

What happens when we solve $y = \sqrt{x}$ for x?

$y = \sqrt{x}, \ y \geq 0$

$y^2 = x, \ y \geq 0$

$x^2 = y, \ x \geq 0$

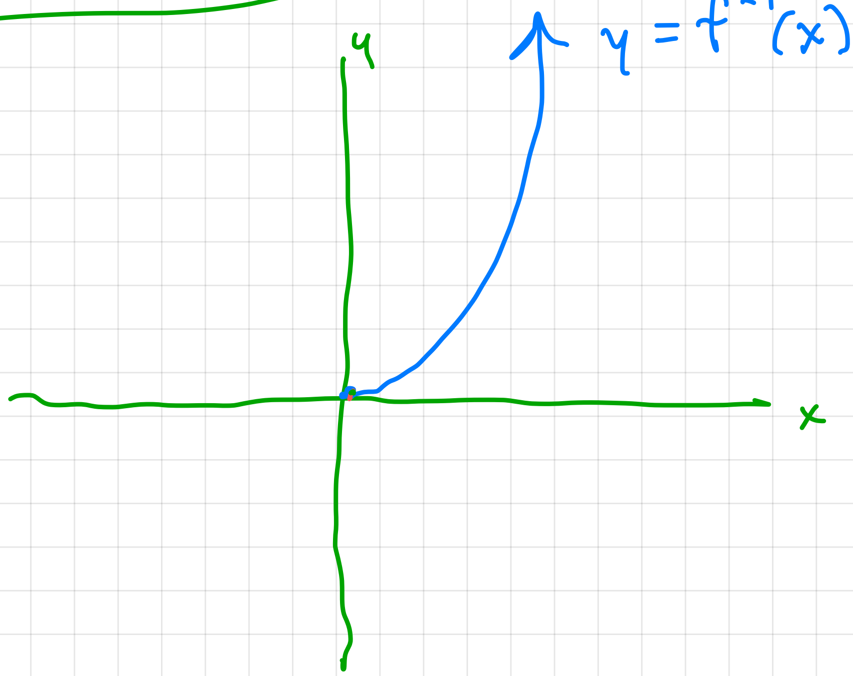


Summary of this example

$f(x) = \sqrt{x}$ does have an inverse function.

$$f^{-1}(x) = x^2, \quad x \geq 0$$

It's graph is the right half of the parabola $y = x^2$. By removing the left side of the parabola we get a curve that does satisfy HLP.



conclude

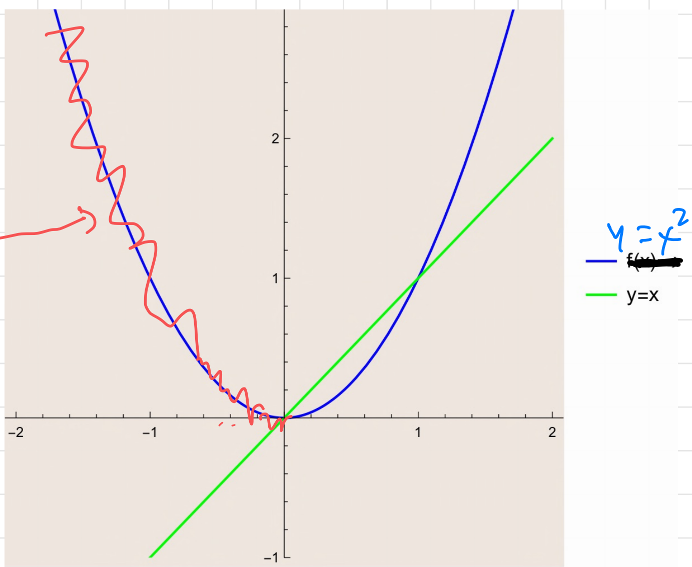
The function

$$g(x) = x^2, \; x \geq 0$$

does have an inverse function, and

$$g^{-1}(x) = \sqrt{x}$$

remove this

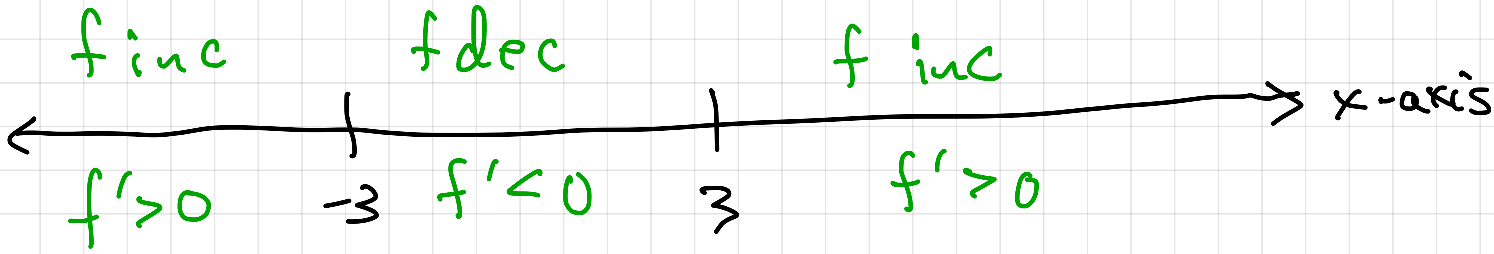


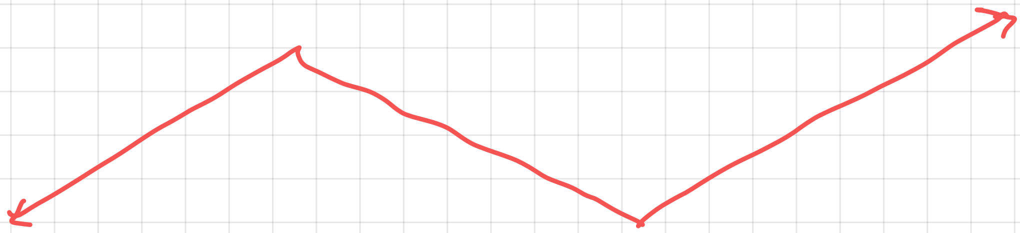
④ $f(x) = x^3 - 27x + 10$ $\quad$ $f'(x) = 3x^2 - 27$

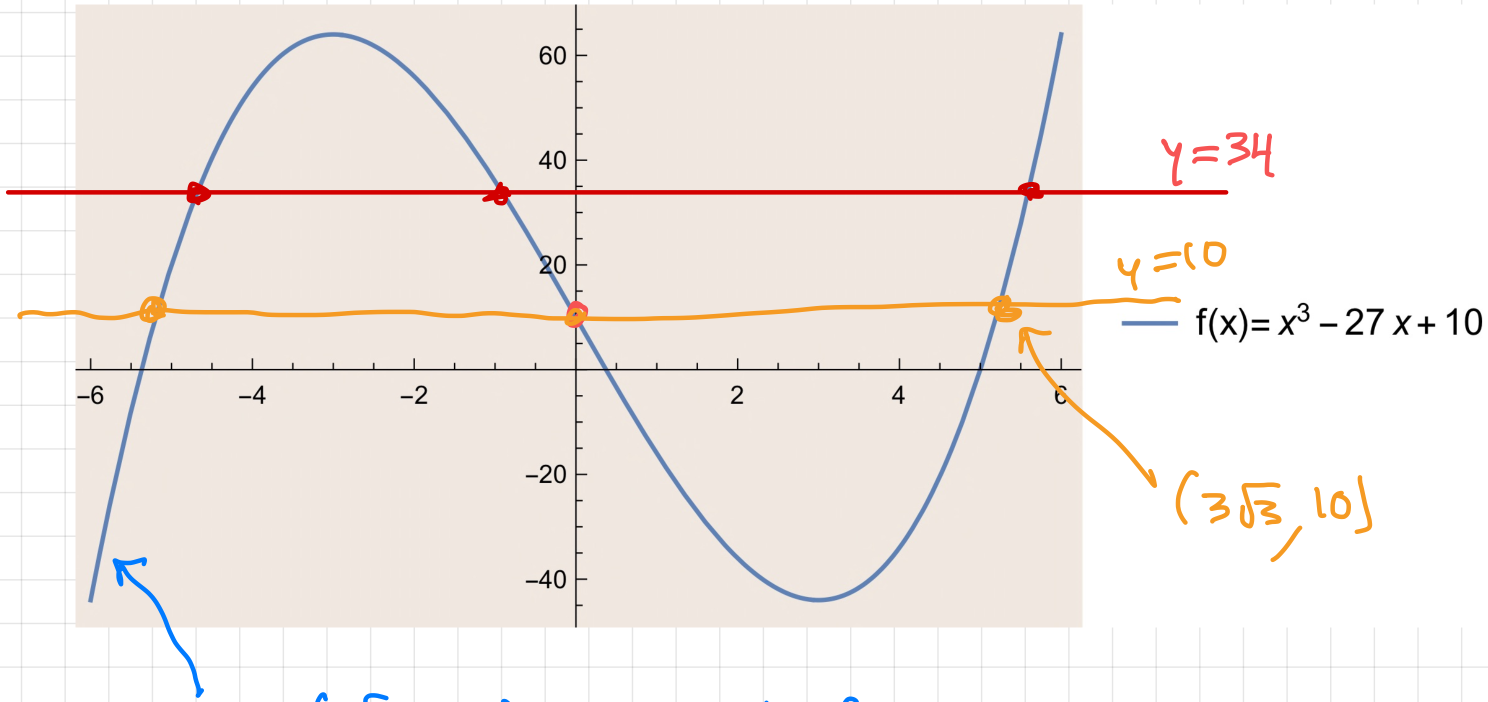
Use calculus to sketch the graph $y = f(x)$ . . . .

critical #'s $\quad 3x^2 - 27 = 0 \iff 3(x-3)(x+3) = 0$

$\Rightarrow x = 3$ and $x = -3$ are critical numbers







Graph of f(x) doesn't satisfy HLP, so there is no inverse function.

You could also observe, for example, that

$$f(0) = f(3\sqrt{3}) = f(-3\sqrt{3}) = 10$$

but $0 \neq 3\sqrt{3} \neq -3\sqrt{3}$

from Stewart page 400-401:

2 Definition Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

and

1 Definition A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

So, if $f(x)$ is one-to-one then $f^{-1}(x)$ is the number y for which $f(y)=x$.

Definition of natural logarithm function

$$\boxed{\text{For } x>0, \quad \ln(x) = \int_1^x \frac{1}{t}\,dt}$$

$$\boxed{\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}}, \quad x>0$$

$$\boxed{\int \frac{1}{x}\,dx = \ln(x) + C, \quad x>0}$$

First Observations:

- $\text{domain}(\ln) = (0,\infty) = \{x \text{ where } x>0\}$
- $\text{range}(\ln) = \mathbb{R} = (-\infty,\infty)$
- $$\ln(x) = \begin{cases} <0 & \text{when } 0<x<1 \\ =0 & \text{when } x=1 \\ >0 & \text{when } x>1 \end{cases}$$
- The function $f(x)$ is increasing and concave down for all $x>0$. ← why?

ALSO

why?

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C, \quad x\neq 0}$$

BECAUSE:

For $x<0$, $\quad \frac{d}{dx}\left[\ln(\underbrace{-x}_{-x>0})\right] = \frac{1}{-x}\frac{d}{dx}[-x] = \frac{1}{x}$

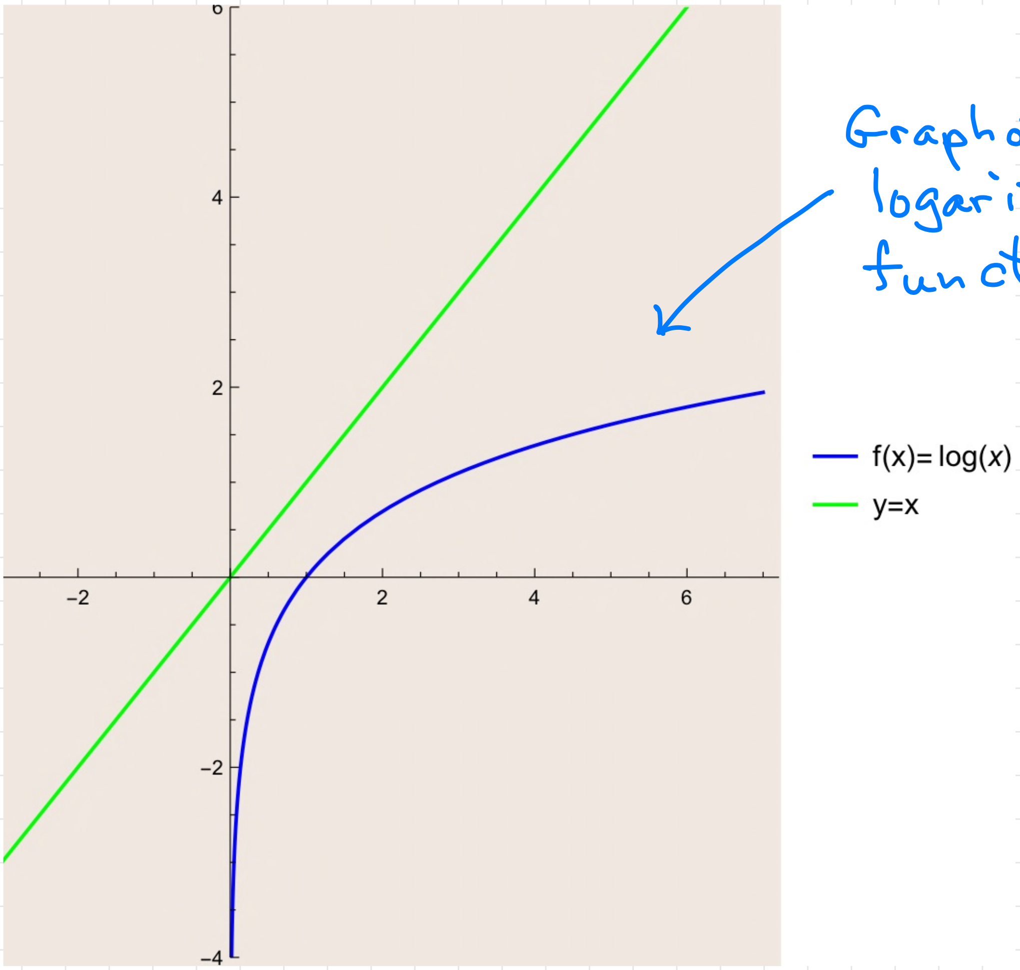
Graph of $f(x) = \ln(x)$ $(x>0)$

$f'(1) = \frac{1}{1} = 1 > 0$

- $f'(x) = 1/x > 0 \Rightarrow f$ has no critical numbers

  $\Rightarrow \ln(x)$ is strictly increasing
- $f''(x) = -\frac{1}{x^2} < 0 \Rightarrow f'$ has no critical numbers

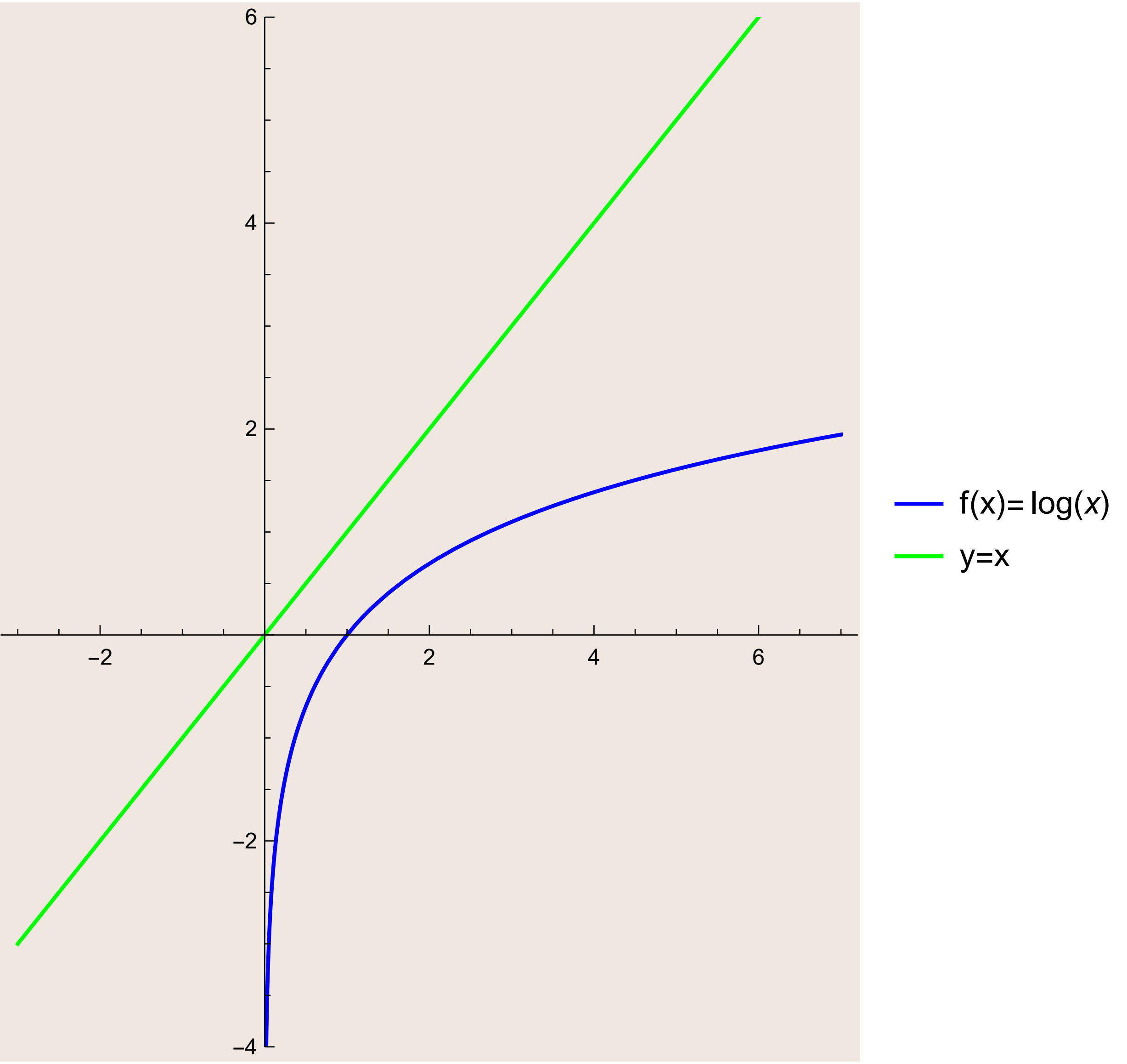
  $\Rightarrow y=\ln(x)$ is concave down



Graph of natural logarithm function.

Observe:

- $f(x) = \ln(x)$ is one-to-one.
- Graph of $y=\ln(x)$ is below $y=x$.
- y-axis is a vertical asymptote for $y=\ln(x)$
- $y=\ln(x)$ has no horizontal asymptote



f(x)=log(x)
y=x
6
4
2
−2
−4
−2
2
4
6

Logarithms – Algebraic Properties

① Suppose $x>0$ and $a>0$ then

$$\frac{d}{dx}\left[\ln(ax)\right] = \frac{1}{ax}\cdot a = \frac{1}{x} = \frac{d}{dx}\left[\ln(x)\right]$$

$$\Rightarrow \ln(ax) = \ln(x) + C. \quad (*)$$

Since $\ln(a) = \ln(a\cdot 1) = \ln(1) + C = C$, equation $(*)$ says $\ln(ax) = \ln(a) + \ln(x)$.

② If $x>0$ and p is a rational number then

$$\frac{d}{dx}\left[\ln(x^p)\right] = \frac{1}{x^p}\cdot p x^{p-1} = \frac{p}{x} = \frac{d}{dx}\left[p\ln(x)\right]$$

$$\Rightarrow \ln(x^p) = p\ln(x) + C \quad (**)$$

Plugging $x=1$ into $(**)$ gives $C=0$ so

$$\ln(x^p) = p\ln(x)$$

Conclusions

powerful result: ln converts multiplication to addition.

① $\ln(a\cdot b) = \ln(a) + \ln(b)$ for $a>0, b>0$

② $\ln(a^p) = p\ln(a)$ for $a>0$ and p rational

example $\ln\left(\frac{1}{x}\right) = \ln(x^{-1}) = -\ln(x)$

for $x>0$