Important Notation Usage

True or False?

① $x = 3 \Rightarrow x^2 = 9$ T

② $x^2 = 9 \Rightarrow x = 3$ F

③ $x = 16 \Rightarrow \sqrt{x} = 4$ T

④ $x^2 = 9 \Leftrightarrow x = \pm 3$ T

⑤ $f(1) = 7$ and $f(2) = 7 \Rightarrow f$ is not 1-1 T

⑥ $3/7 = .428571$ F

⑦ $\pi \approx 3.1415$ T

⑧ $\pi \approx 100$ T

"$\Rightarrow$" or "$\rightarrow$" denotes implication.

EG $x=3 \Rightarrow x^2=9$ means

- "$x=3$ implies that $x^2=9$." or
- "If $x=3$ then $x^2=9$."

"$\Leftrightarrow$" or "$\leftrightarrow$" denotes logical equivalence

EG "$x=\pm 3 \Leftrightarrow x^2=9$". means

- Both "$x=\pm 3 \Rightarrow x^2=9$" and "$x^2=9 \Rightarrow x=\pm 3$" are true
- "$x=\pm 3$ if and only if $x^2=9$"

"$\approx$" or "$\sim$" denotes approximation.

Always remember that 'approximation' is a relative concept.

Common Comments from Exam 1

The goal is to draw good schematic graphs to use to aid in solving problems.

Work on making your graphs more robust and accurate.

A little hard to follow your logic here. Try to work on organizing your explanations more clearly.

Work on making your logic more clear. This will be very important as problems get more intricate.

equal? Then you must write "=".

Does "$\longrightarrow$" mean "equals"? Write "=" then!

0 and 1 are limits for x, not for u.

Example from last class revisited:

The two functions

$$f(x) = x^2$$

$$g(x) = x^2, \quad x \geq 0$$

are different functions! Because

$$\text{domain}(f) = \mathbb{R}$$

but

$$\text{domain}(g) = [0, \infty)$$

<u>The function f(x)</u>: It's graph does not satisfy HLP, and $f(x)$ is <u>not</u> one-to-one. It has no inverse function.

<u>The function g(x)</u>: It's graph does satisfy HLP, and $g(x)$ <u>is</u> one-to-one. It has an inverse function and

$$g^{-1}(x) = \sqrt{x}, \quad x \geq 0$$

1 Definition A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

abbreviate:

$1\text{-}1 \equiv$ one-to-one

$\boxed{\text{One-to-one functions are the functions which have inverse functions.}}$

Fact: If $f(x)$ is differentiable and has an inverse function $f^{-1}(x)$ then $f^{-1}(x)$ is also differentiable.

To find $\frac{d}{dx}\left[f^{-1}(x)\right]$ use the formula $f(f^{-1}(x)) = x$:

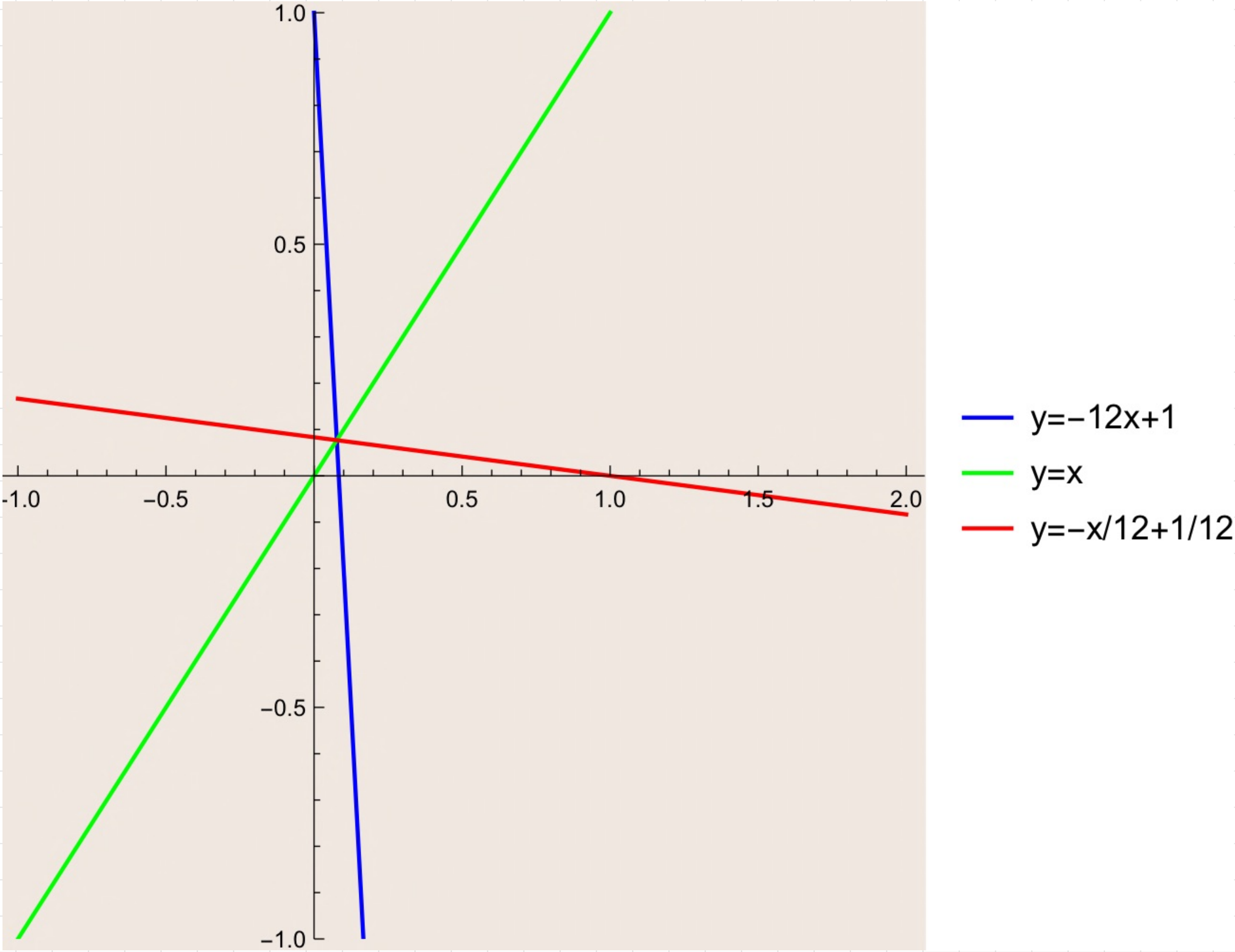
$$\frac{d}{dx}\left[f(f^{-1}(x))\right] = f'(f^{-1}(x))\,\frac{d}{dx}\left[f^{-1}(x)\right]$$

$$\frac{d}{dx}\left[f(f^{-1}(x))\right] = \frac{d}{dx}[x] = 1$$

$$\Rightarrow \qquad \frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}$$

① Example

Graphs of $f(x) = -12x + 1$ and $f^{-1}(x) = -\frac{1}{12}x + \frac{1}{12}$:



$f'(x) = -12$, $(f^{-1})'(x) = \frac{1}{-12}$

Definition of natural logarithm function

$$\boxed{\text{For } x>0, \quad \ln(x) = \int_1^x \frac{1}{t}\,dt}$$

$$\boxed{\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}}\ , \ x>0$$

$$\boxed{\int \frac{1}{x}\,dx = \ln(x) + C, \quad x>0}$$

First Observations:

- $\text{domain}(\ln) = (0,\infty) = \{x \text{ where } x>0\}$
- $\text{range}(\ln) = \mathbb{R} = (-\infty,\infty)$
- $$\ln(x) = \begin{cases} <0 & \text{when } 0<x<1 \\ =0 & \text{when } x=1 \\ >0 & \text{when } x>1 \end{cases}$$
- The function $f(x)$ is increasing and concave down for all $x>0$. ← why?

ALSO

why?

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C, \quad x\neq 0}$$

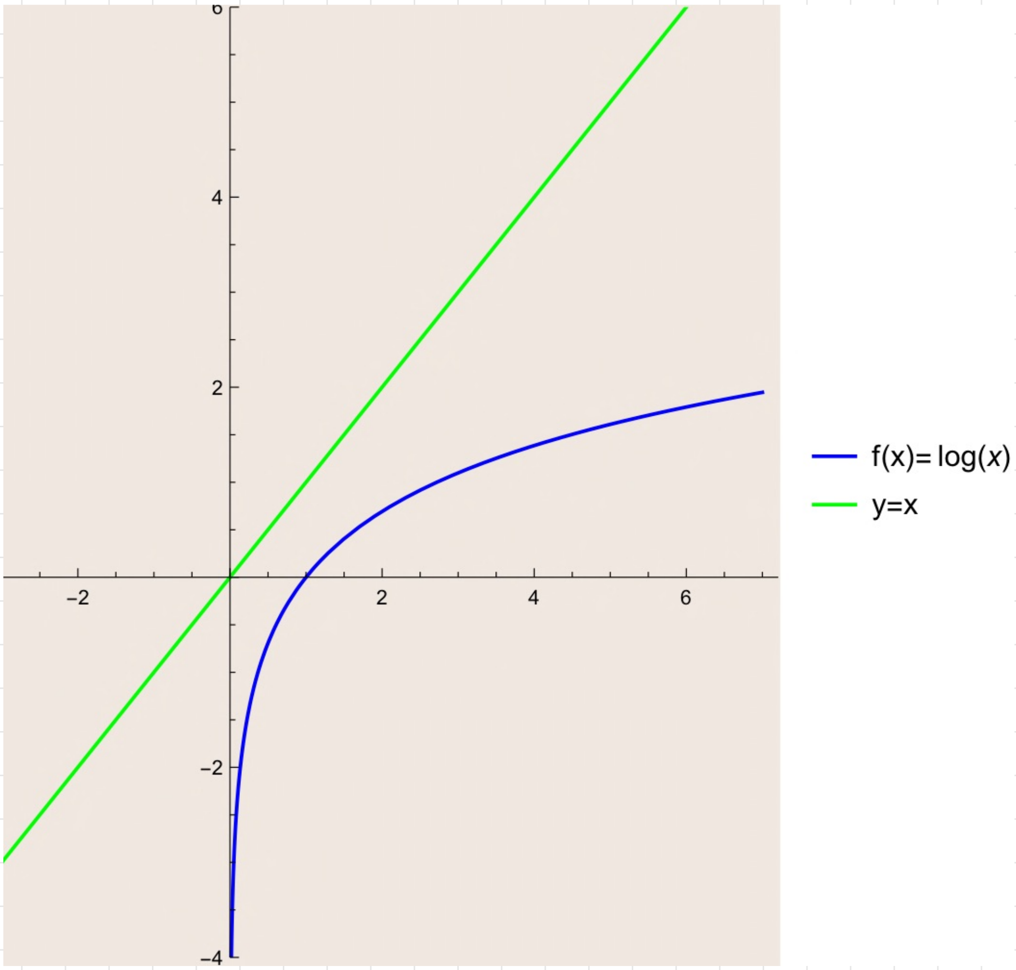
BECAUSE:

For $x<0$, $\frac{d}{dx}\left[\ln(-x)\right] = \frac{1}{-x}\frac{d}{dx}[-x] = \frac{1}{x}$ $\quad -x>0$

Graph of $f(x) = \ln(x)$ $(x>0)$

$f'(1) = \frac{1}{1} = 1 > 0$

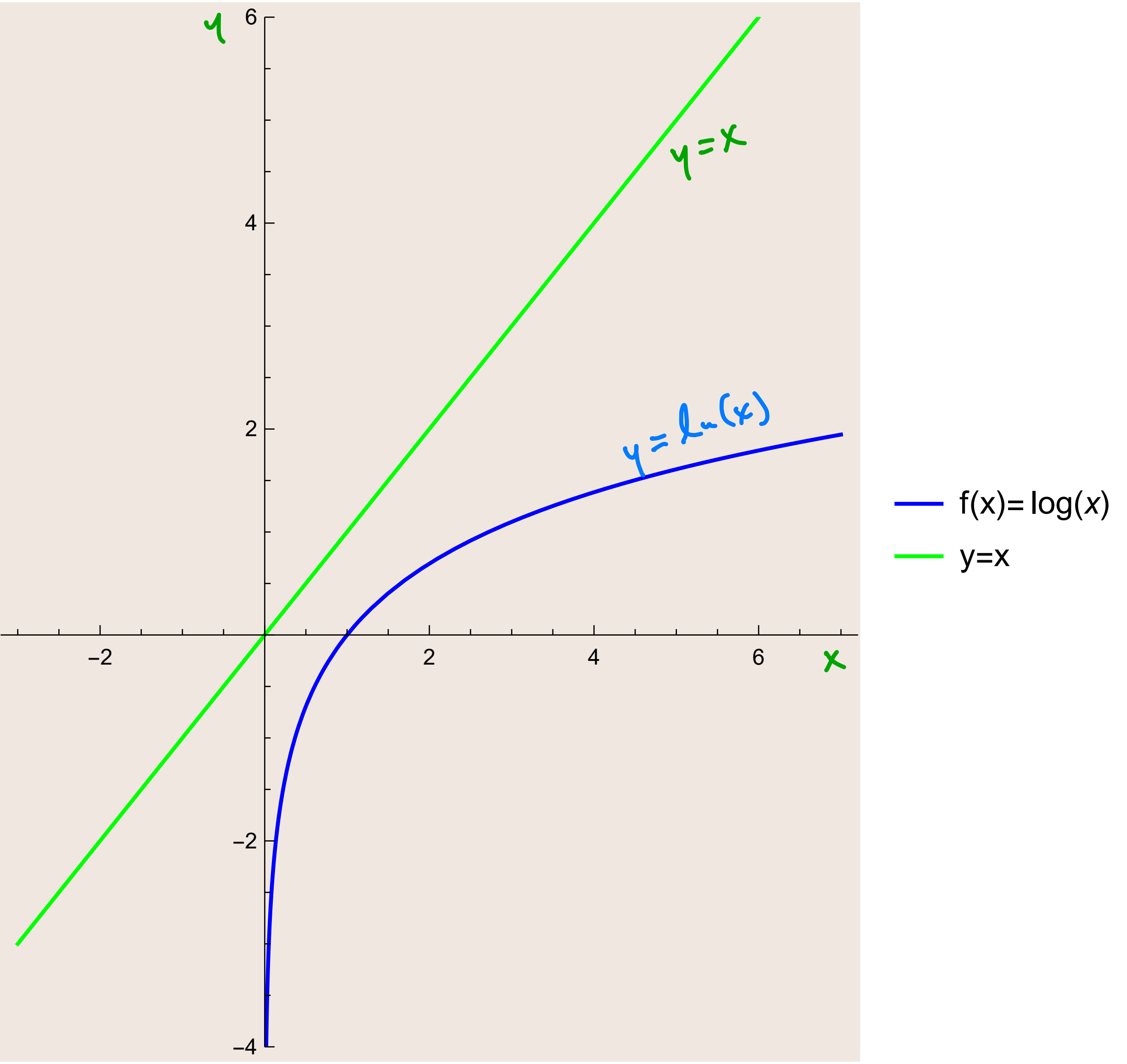
- $f'(x) = 1/x > 0 \Rightarrow f$ has no critical numbers
  $\Rightarrow \ln(x)$ is strictly increasing
- $f''(x) = -\frac{1}{x^2} < 0 \Rightarrow f'$ has no critical numbers
  $\Rightarrow \ln(x)$ is concave down



Observe:

$\lim_{x\to 0+} \ln(x) = -\infty$

- $f(x) = \ln(x)$ is one-to-one.
- Graph of $y = \ln(x)$ is below $y = x$.
- y-axis is a vertical asymptote for $y = \ln(x)$
- $y = \ln(x)$ has no horizontal asymptote



y
6
4
2
−2
−4
−2
2
4
6
x
y=x
y=ln(x)
f(x)=log(x)
y=x

Logarithms – Algebraic Properties

① Suppose $x>0$ and $a>0$ then

$$\frac{d}{dx}\left[\ln(ax)\right] = \frac{1}{ax}\cdot a = \frac{1}{x} = \frac{d}{dx}\left[\ln(x)\right]$$

$$\Rightarrow \ln(ax) = \ln(x) + C. \qquad (*)$$

Since $\ln(a) = \ln(a\cdot 1) = \ln(1) + C = C$, equation $(*)$ says $\ln(ax) = \ln(a) + \ln(x)$.

② If $x>0$ and p is a rational number then

$$\frac{d}{dx}\left[\ln(x^p)\right] = \frac{1}{x^p}\cdot p x^{p-1} = \frac{p}{x} = \frac{d}{dx}\left[p\ln(x)\right]$$

$$\Longrightarrow \ln(x^p) = p\ln(x) + C \qquad (**)$$

Plugging $x=1$ into $(**)$ gives $C=0$ so

$$\ln(x^p) = p\ln(x)$$

Conclusions

powerful result

① $\ln(a\cdot b) = \ln(a) + \ln(b)$ for $a>0, b>0$

② $\ln(a^p) = p\ln(a)$ for $a>0$ and p rational

example $\ln\left(\frac{1}{x}\right) = \ln(x^{-1}) = -\ln(x)$

for $x>0$

Problem $f(x) = x^2 - \ln(x)$

Any local extremes? Points of inflection?

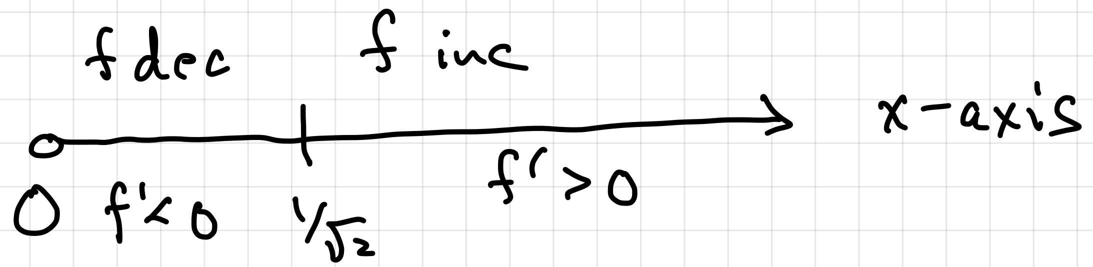
$\text{domain}(f) = (0,\infty) =$ open interval from 0 to ∞

$$f'(x) = 2x - \frac{1}{x}$$

$$f''(x) = 2 + \frac{1}{x^2}$$

local extremes $2x - \frac{1}{x} = 0 \Rightarrow 2x^2 - 1 = 0$

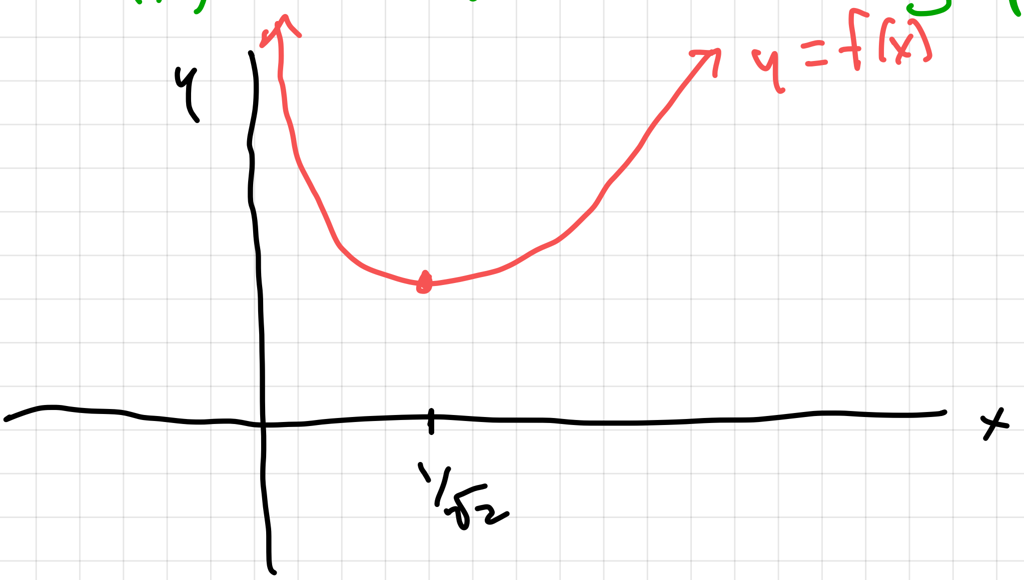
$\Rightarrow 2x^2 = 1 \Rightarrow x = \pm 1/\sqrt{2} \Rightarrow x = 1/\sqrt{2}$



$\Rightarrow$ local min at $x = 1/\sqrt{2}$

points of inflection $2 + \frac{1}{x^2} = 0$ no solutions

$f''(x) > 0$ for all $x > 0 \Rightarrow$ graph of $f(x)$ is concave



$f\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{1}{2} - \ln(2^{-1/2})$

$= \frac{1}{2} + \frac{1}{2}\ln(2)$

$\approx .8466$

Problem

$$g(x) = \ln(x^2+1)$$

Any local extremes? Points of inflection?

next time

The natural logarithm function $\ln(x)$ is strictly increasing so it has an inverse function called the <u>natural exponential function</u> and denoted by $\exp(x)$.

Remembering that

> If $f(x)$ is one-to-one then $f^{-1}(x)$ is the number y for which $f(y)=x$.

we can write:

$$\boxed{\exp(x) \text{ is the number } y \text{ for which } \ln(y)=x.}$$

and

① $\ln(\exp(x))=x$, for any real number x

② $\exp(\ln(x))=x$, for $x>0$

Differentiating ① gives

$$1=\frac{d}{dx}[x]=\frac{d}{dx}[\ln(\exp x)]=\frac{1}{\exp(x)}\exp'(x)$$

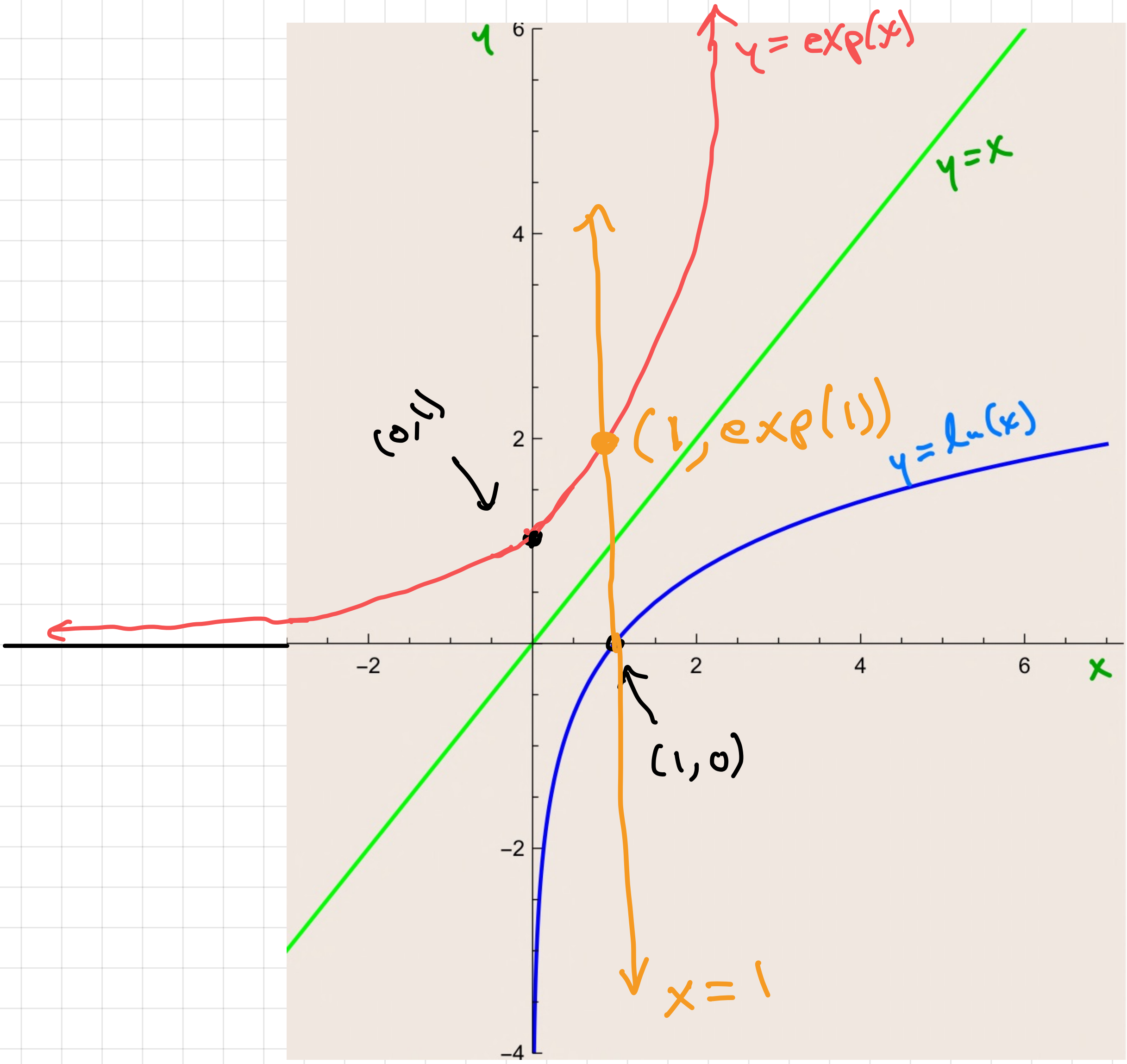
$$\Rightarrow \quad \boxed{\exp'(x)=\exp(x)}$$

So

$$\frac{d}{dx}\left[\exp x\right] = \exp(x)$$

and

$$\int \exp(x)\,dx = \exp(x) + C$$



$\exp(1) = e \approx 2.7818$

- $\exp(0) = 1$
- $\exp(x) > 0$ for all x
- x-axis is a horizontal asymptote on the left
- $0 < \exp(x) < 1$ for $x < 0$
- $\exp(x) > 1$ for $x > 0$.