Exam 2

Next Wednesday, March 24 p

In-Class Portion: 2:00 - 2:45

Written work with pdf to be submitted by 2:55.

must be!

Take-Home Portion:

A set of Webwork problems with at most 3 attempts per problem. Open between

10:00PM on 3/23 and 11:59 on 3/24

There will be a Problem Review Session on Tuesday 3/23, late afternoon/ early evening.

Natural Logarithms – Some Key Properties

$$\ln(x) = \int_1^x \frac{1}{t}\,dt, \quad x>0 \Rightarrow \operatorname{domain}(\ln) = (0,\infty)$$

$$\frac{d}{dx}\left[\ln x\right] = \frac{1}{x} \qquad \int \frac{1}{x}\,dx = \ln|x| + C, \quad x>0$$

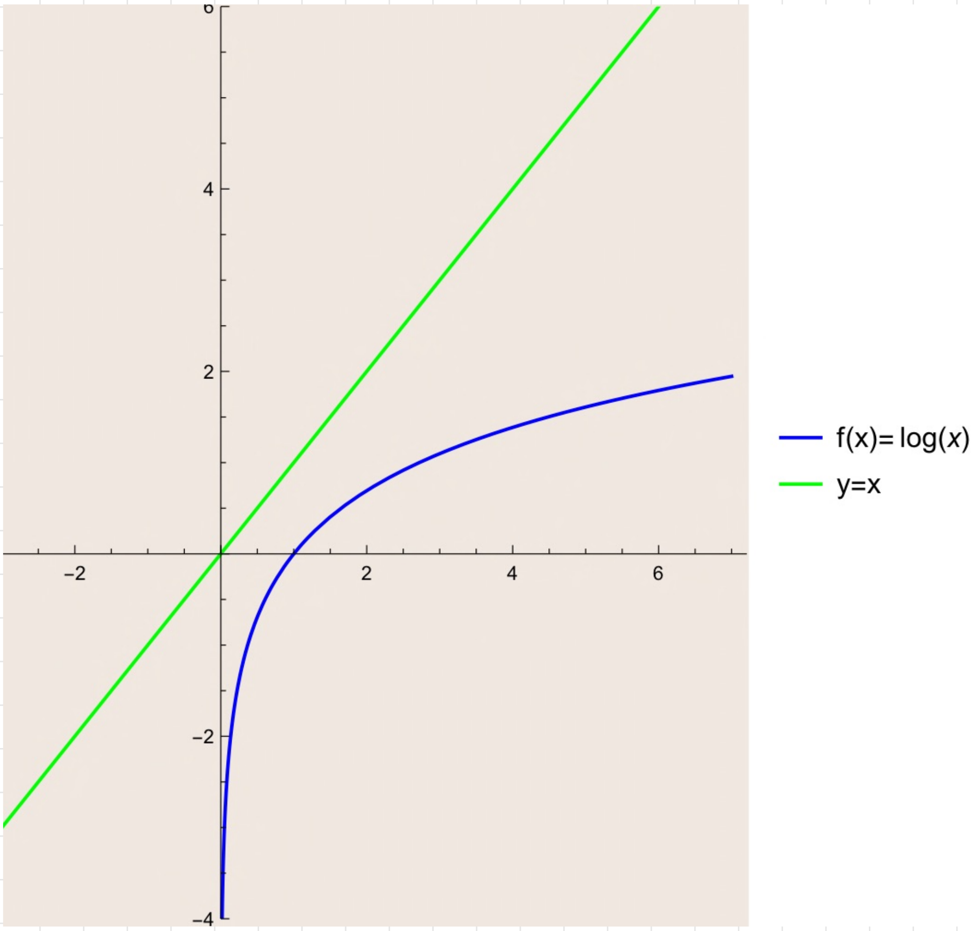
$$\ln(1) = 0, \quad \ln(e) = 1$$

For $a, b > 0$ and p rational

$$\ln(ab) = \ln(a) + \ln(b)$$

$$\ln(a^p) = p\ln(a)$$

(*) see next page



(*)

Algebraic Properties of $\ln(x)$ can be summarized by saying:

- ln converts multiplication to addition
- ln converts exponentiation to multiplication.

(and only this)

These can be very useful properties because addition is "easier" than multiplication and multiplication is "easier" than exponentiation.

Example Find $\frac{d}{dx}\left[(x+1)(x+2)^2(x+3)^3\right]$.

Write $y=(x+1)(x+2)^2(x+3)^3$. Then

$$\ln(y)=\ln(x+1)+2\ln(x+2)+3\ln(x+3)$$

Now differentiate

$$\frac{d}{dx}[\ln y] = \frac{1}{y}\frac{dy}{dx}$$

$$\|$$

$$\frac{d}{dx}\left[\ln(x+1)+2\ln(x+2)+3\ln(x+3)\right]$$

$$= \frac{1}{x+1}+\frac{2}{x+2}+\frac{3}{x+3}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1}{x+1}+\frac{2}{x+2}+\frac{3}{x+3}\right)$$

$$=(x+1)(x+2)^2(x+3)^3\left(\frac{1}{x+1}+\frac{2}{x+2}+\frac{3}{x+3}\right)$$

This process is called "logarithmic differentiation".

Problem $g(x) = \ln(x^2+1)$

Any local extremes? Points of inflection? Graph?

Suggested steps to use in graphing a function: Find each of:

- domain
- critical #'s for $g(x)$
- increase/decrease
- critical #'s for $g'(x)$
- concavity
- plug in important points and draw graph.
- special properties

Problem $g(x) = \ln(x^2+1)$

Any local extremes? Points of inflection? Graph?

- **Domain** $\text{domain}(g) = \{x \text{ where } x^2+1>0\} = \mathbb{R} = (-\infty,\infty)$

- **Critical #'s for g** $g'(x) = \dfrac{1}{x^2+1}\dfrac{d}{dx}[x^2+1] = \dfrac{2x}{x^2+1}$

$\dfrac{2x}{x^2+1} = 0 \iff 2x=0 \iff x=0$ ← only critical #

- **Inc/Dec**

local min

	g dec		g inc	
←	$g'(x)<0$	0	$g'(x)>0$	→ x

- **Critical #'s for g'** $g''(x) = \dfrac{2(x^2+1)-2x(2x)}{(x^2+1)^2} = \dfrac{2-2x^2}{(x^2+1)^2}$

$\dfrac{2-2x^2}{(x^2+1)^2} = 0 \iff 2-2x^2=0 \iff 2(1-x)(1+x)=0$

$\iff x=1$ or $x=-1$ ← only critical numbers for g'

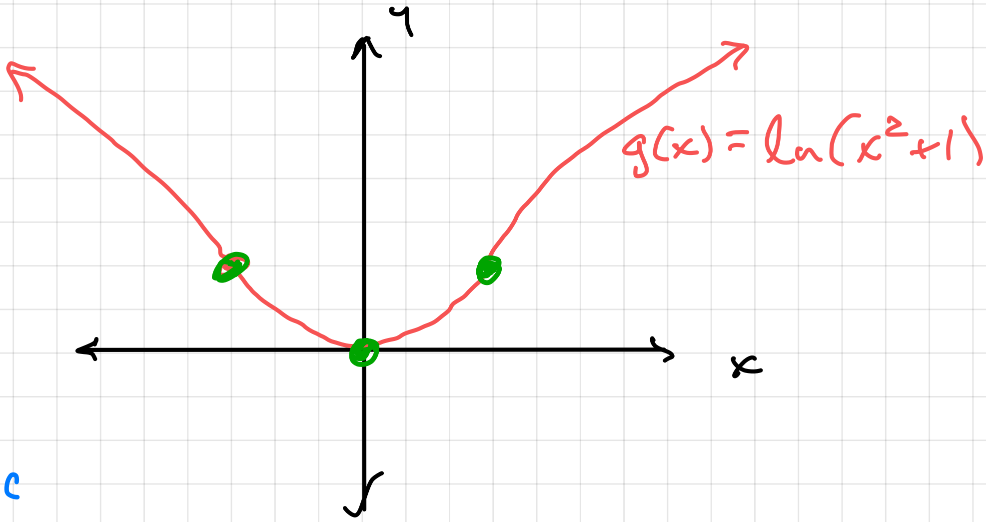
- **concavity**

	c. down		c. up		c. down	
←	$g''(x)<0$	-1	$g''(x)>0$	1	$g''(x)<0$	→ x

points of inflection

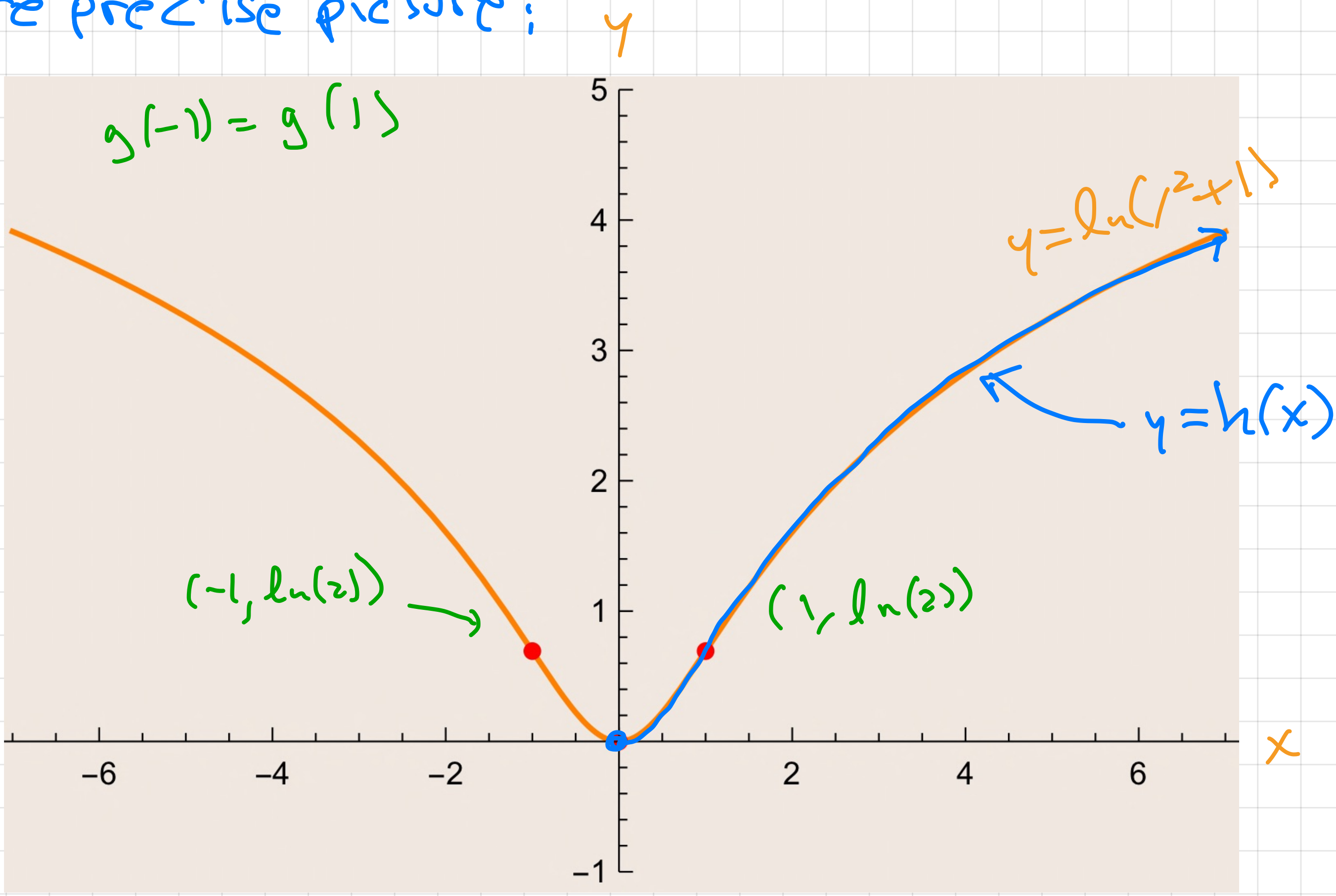
- **rough graph**

x	g(x)	
0	0	$=\ln(1)$
-1	$\ln(2) \sim .69$	
1	$\ln(2) \sim .69$	



A crude but reliable schematic picture

More precise picture:



note $g(x) = \ln(x^2+1)$ is an even function

because $g(-x) = \ln((-x)^2+1) = \ln(x^2+1) = g(x)$.

note $g(x) = \ln(x^2+1)$ is not 1−1

(and so it doesn't have an inverse function)

However, $h(x) = \ln(x^2+1)$, $x \geq 0$

does have an inverse.

$y = \ln(x^2+1)$, $x \geq 0$ $\Rightarrow y \geq 0 = \ln(1)$

$\exp(y) = \exp(\ln(x^2+1)) = x^2+1$, $x \geq 0$

$x^2 = \exp(y) - 1$, $x \geq 0$ $x \geq 0$

$x = \sqrt{\exp(y)-1}$, $x \geq 0$

$f^{-1}(x) = (\exp(x)-1)^{1/2}$, $x \geq 0$

Problem

$g(x) = \ln(x^2+1)$

Find absolute max and min for g where $-3 \le x \le 2$,

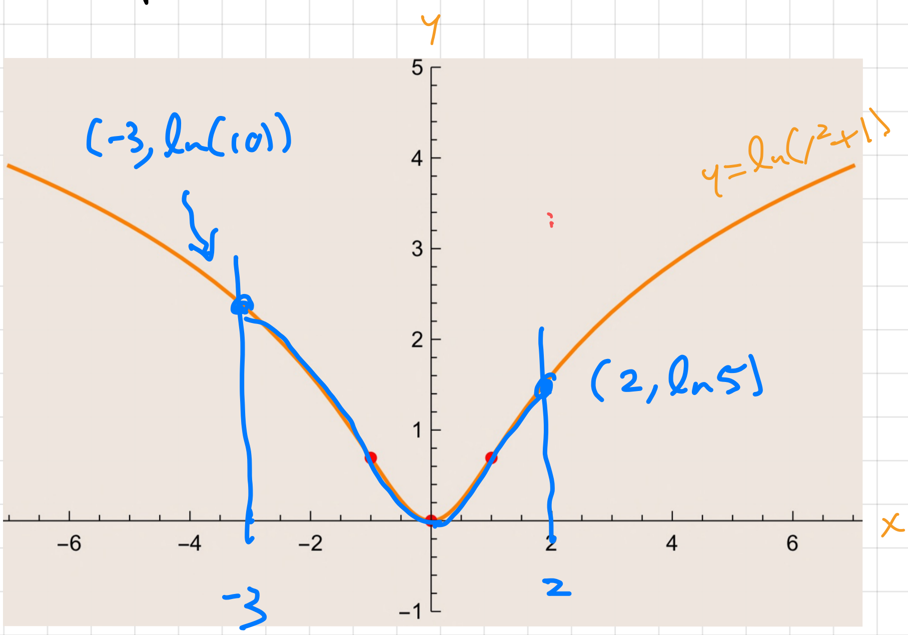
$\boxed{\text{KNOW: Absolute extremes occur either at critical \#'s inside } [-3,2] \text{ or at the endpoints } -3 \text{ and } 2.}$

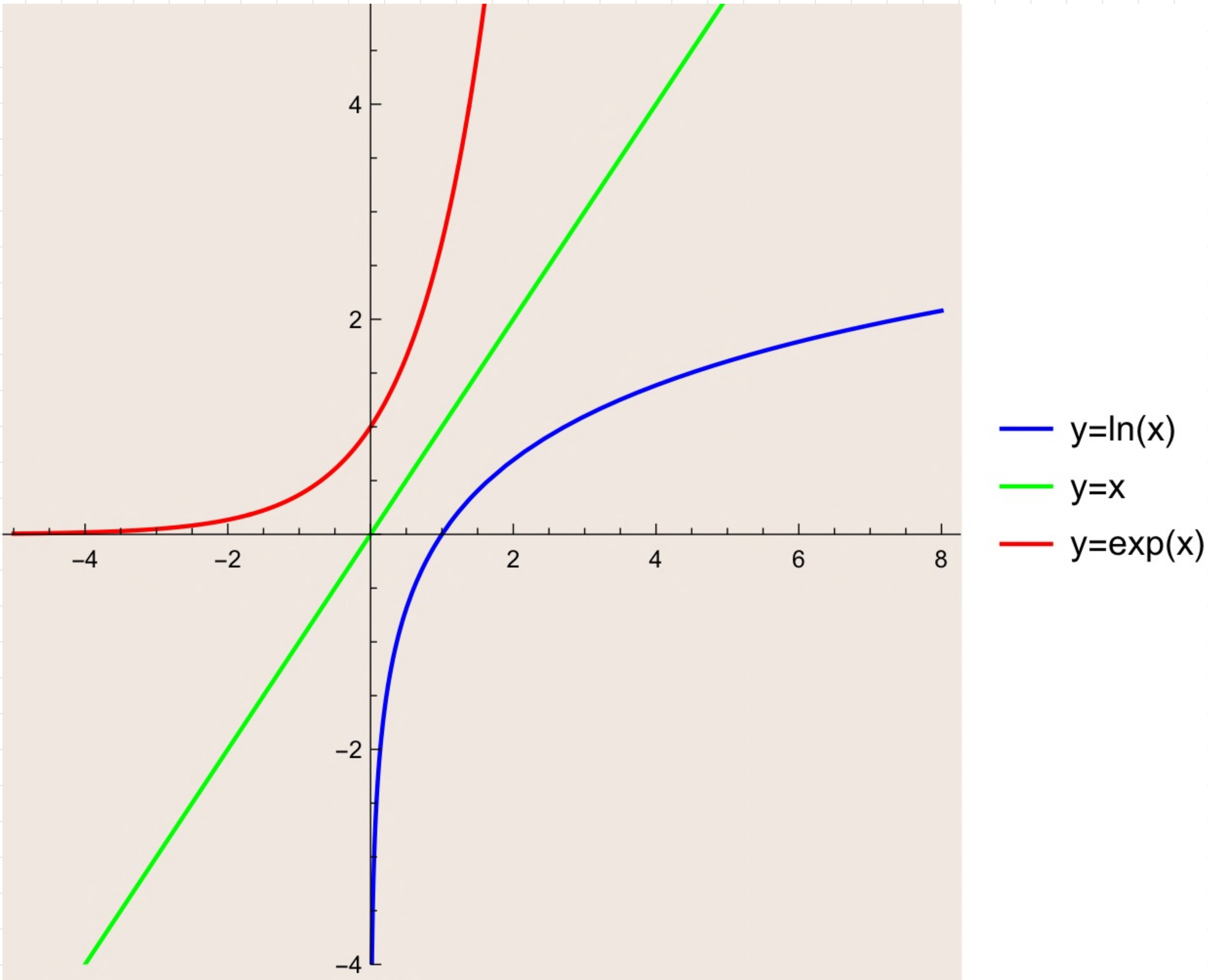
$x=0$ is only critical # and it is inside $[-3,2]$.

x	$g(x)$	
0	0	← min
-3	$\ln(10)$	← max
2	$\ln(5)$	

$\ln(10) > \ln(5)$ because $\ln(x)$ is increasing.

Or you can answer the question by refering to a reliable graph:





Natural Exponential Function

exp(x) is the inverse function of ln(x)

$\text{domain}(\exp) = \mathbb{R} = \text{range}(\ln)$

$\text{range}(\exp) = (0, +\infty) = \text{domain}(\ln)$

<u>Definition</u> $\exp(x)$ is the inverse function of $\ln(x)$.

$$\boxed{\exp(x) \text{ is the number } y \text{ for which } \ln(y) = x.}$$

<u>E.G.</u> $\ln(1) = 0 \Rightarrow \exp(0) = 1$

$\ln(e) = 1 \Rightarrow \exp(1) = e$

$$e = \exp(1) \approx 2.71828182846$$

$$\ln(\exp(x)) = x, \text{ for any real number } x$$

$$\exp(\ln(x)) = x, \text{ for } x > 0$$

$$\frac{d}{dx}\left[\exp(x)\right] = \exp(x)$$

$$\int \exp(x)\,dx = \exp(x) + C$$

Take $a=1$ here

$$\exp(a+b) = \exp(a)\exp(b)$$

$$\exp(pa) = (\exp(a))^p, \ p \text{ rational}$$

<u>NOTE</u>: $\exp(p) = \exp(p \cdot 1) = \exp(1)^p = e^p$ for p rational

NOTE: $\exp(p) = \exp(p \cdot 1) = \exp(1)^p = e^p$ for p rational

With this it makes sense to define

$$e^x = \exp(x)$$

for any real number.

So these are two notations for the same function.

Recap of properties using new notation

$\boxed{e^x \text{ is the number } y \text{ for which } \ln(y) = x.}$

$\ln(e^x) = x$, for any real number x

$e^{\ln(x)} = x$, for $x > 0$

$$\frac{d}{dx}\left[e^x\right] = e^x$$

$$\int e^x \, dx = e^x + C$$

$$e^{a+b} = e^a e^b$$

$$e^{ab} = (e^a)^b$$

Laws of exponents.

General Exponential Functions

If $b>0$ and p is rational then

$$\ln(b^p) = p\ln(b)$$

$$\Rightarrow \quad b^p = \exp(\ln b^p) = e^{p\ln(b)}$$

So <u>define</u> b^x for any number x and $b>0$ by

$$\boxed{b^x = e^{x\ln(b)}}$$

Then

$$\frac{d}{dx}\left[b^x\right] = \frac{d}{dx}\left[e^{x\ln(b)}\right] = e^{x\ln(b)}\frac{d}{dx}\left[x\ln(b)\right]$$

$$= \ln(b)\, b^x$$

And

$$\int b^x\,dx = \frac{1}{\ln(b)}\, b^x$$

Find $f'(x)$

① $f(x) = \ln(x)e^x$

② $f(x) = e^{e^{e^x}}$

③ $f(x) = \ln(2)$

Integrate

③ $\int x^3 e^{x^4}\,dx$

④ $\int (e^x + e^{-x})^2\,dx$

for next class...