Exam 2 : Wednesday, March 24

In-Class Portion : 2:00 - 2:45

Written work with pdf to be submitted by 2:55. must be!

Must be logged on to Zoom with video camera on, audio turned off, keep an eye on chat. No headphones or ear buds. OU picture ID at hand.

Take-Home Portion : A set of Webwork problems with at most 3 attempts per problem. Open between

10:00PM on 3/23 and 11:59 on 3/24

4 - 6:30 PM

There will be a Problem Review Session on Tuesday 3/23, late afternoon/ early evening.

recap of recent classes:

6.1 ← Pairs of inverse functions

6.2* ← natural log function

6.3* ← natural exponential function

6.4* ← logs and exponentials with base b

$$\ln(x) = \int_1^x \frac{1}{t}\,dt, \quad x>0$$

$\exp(x) = e^x =$ inverse function of $\ln(x)$

note: The natural log and exponential functions have base $b=e$ where $e \approx 2.781828$ satisfies that $\ln(e)=1$.

Stewart p. 446

47–48 Find an equation of the tangent line to the curve at the given point.

48. $y = \ln(x^3 - 7)$, $(2, 0)$

graph of the function
$f(x) = \ln(x^3 - 7)$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\ln(x^3-7)\right] = \frac{1}{x^3-7}\frac{d}{dx}\left[x^3-7\right]$$

$$= \frac{3x^2}{x^3-7}$$

$$\left.\frac{dy}{dx}\right|_{x=2} = \text{slope of desired tangent line}$$

$$= \frac{3(2)^2}{2^3-7} = 12$$

Tangent line goes thru $(2,0)$, so it has equation:

$$y - 0 = 12(x-2)$$

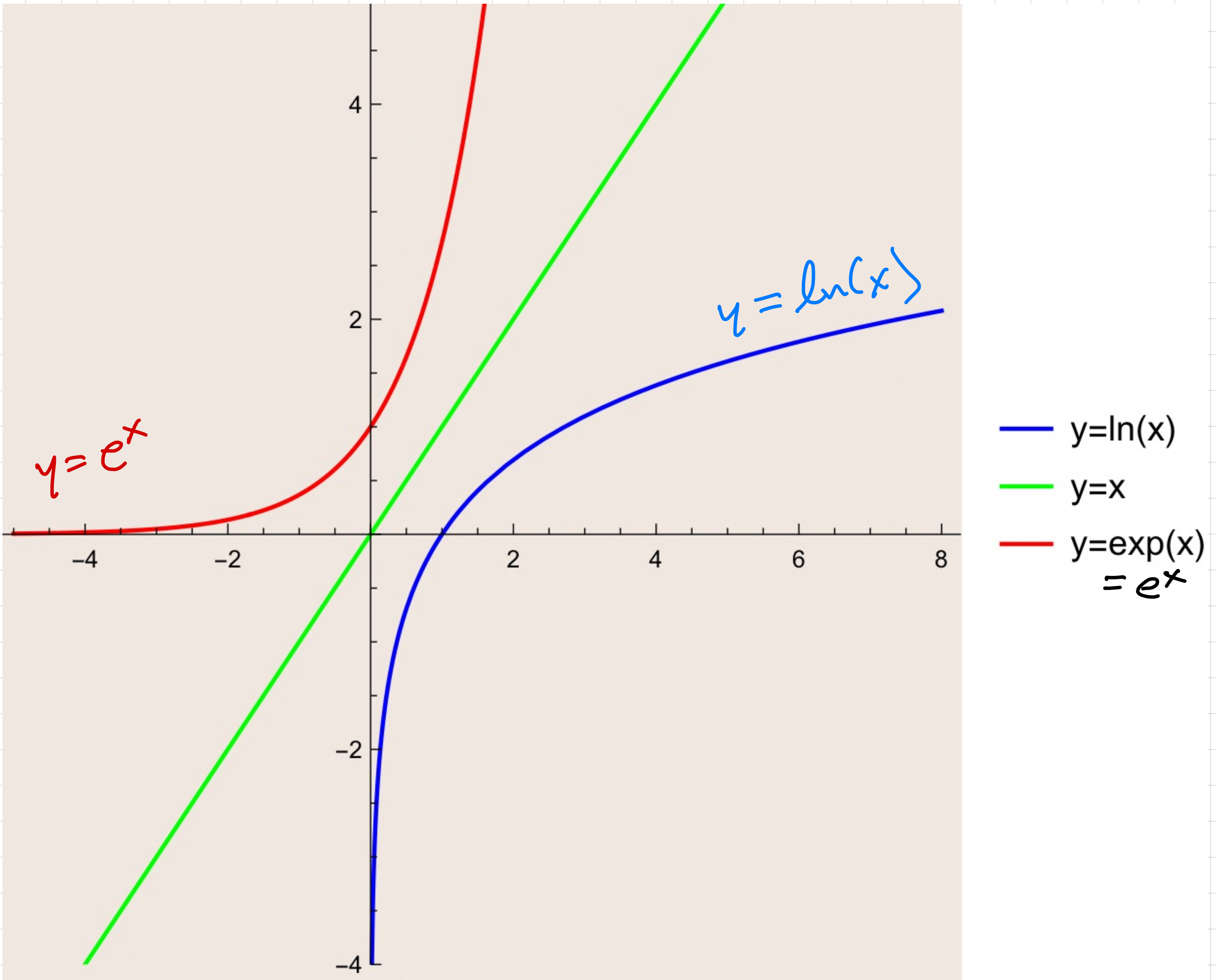
$$y = 12x - 24$$

Natural log and exponential functions

- $\ln(x) = \int_1^x \frac{1}{t}\,dt, \quad x > 0 \Rightarrow \text{domain}(\ln) = (0, \infty)$
- $\frac{d}{dx}\left[\ln x\right] = \frac{1}{x}$
- $\int \frac{1}{x}\,dx = \ln|x| + C, \ x > 0$
- $\ln(1) = 0, \quad \ln(e) = 1$
- $\ln(ab) = \ln(a) + \ln(b)$, for $a, b > 0$
- $\ln(a^p) = p\ln(a)$, for $a > 0$

- $e^x = \exp(x) =$ number y for which $\ln(y) = x$
- $\ln(x) =$ number y for which $e^y = x$, $x > 0$
- $\ln(\exp(x)) = x$, for any real number x
- $\exp(\ln(x)) = x$, for $x > 0$

- $\frac{d}{dx}\left[e^x\right] = e^x$
- $\int e^x\,dx = e^x + C$
- $e^0 = \exp(0) = 1$
- $e^{a+b} = e^a e^b$ $\leftarrow$ Laws of exponents.
- $e^{ab} = (e^a)^b$



$$\mathrm{domain}(\exp) = \mathbb{R} = \mathrm{range}(\ln)$$

$$\mathrm{range}(\exp) = (0, +\infty) = \mathrm{domain}(\ln)$$

Always remember:

$$e^{\text{any power}} > 0$$

EG $e^{-1000} > 0$, $e^{7x^3 + \ln x + 5} > 0$, ...

exponential functions vs. power functions.

For $b>0$ constant

$$f(x) = x^b \quad \text{(power function)}$$

$$g(x) = b^x \quad \text{(exponential)}$$

$$\Rightarrow \quad f'(x) = b x^{b-1}, \quad g'(x) = \ln(b)\, b^x$$

These are very different functions !

$$b^x = e^{x\ln(b)}$$

Calculate $g'(x)$:

$$g'(x) = \frac{d}{dx}\left[b^x\right] = \frac{d}{dx}\left[e^{x\ln(b)}\right]$$

$$= e^{x\ln(b)} \frac{d}{dx}\left[x\ln(b)\right]$$

$$= e^{x\ln(b)} \ln(b) = \ln(b)\, b^x$$

b^x = exponential function with base $b>0$

Find $f'(x)$

$\exp(x) = e^x$

① $f(x) = \ln(x)e^x$

$$f'(x) = \frac{1}{x}e^x + \ln(x)e^x = e^x\left(\frac{1}{x} + \ln x\right)$$

② $f(x) = e^{e^{e^x}} = f(x) = \exp(\exp(\exp x))$

$$f'(x) = \exp(\exp(\exp(x))) \cdot \frac{d}{dx}\left[\exp(\exp(x))\right]$$
$$= e^{e^{e^x}} \cdot \exp(\exp(x)) \cdot \frac{d}{dx}[\exp x]$$
$$= e^{e^{e^x}} e^{e^x} e^x$$

③ $f(x) = \ln(e^x + 1)$

$$f'(x) = \frac{1}{e^x+1}\frac{d}{dx}[e^x+1] = \frac{e^x}{e^x+1}$$

$$f''(x) = \frac{e^x(\cancel{e^x}+1) - \cancel{e^x e^x}}{(e^x+1)^2} = \frac{e^x}{(e^x+1)^2}$$

Does $f(x)$ have any critical points?

No, because $e^x/(e^x+1) > 0$ for all x.

Does $f'(x)$ have any critical points?

No, because $e^x/(e^x+1)^2 > 0$ for all x.

Shows graph of $y = f(x)$ is always increasing and concave up.

④ $f(x) = \ln(2)$, $f'(x) = 0$

because $\ln(2)$ is a constant.

⑤ $\int x^3 e^{x^4}\,dx$ substitute $\begin{cases} u = x^4 \\ du = 4x^3\,dx \end{cases}$

$= \frac{1}{4}\int e^u\,du = \frac{1}{4}e^u + C$

$= \frac{1}{4}e^{x^4} + C$

⑥ $\int (e^x + e^{-x})^2\,dx = \int e^{2x} + 2 + e^{-2x}\,dx$

$= \int e^{2x}\,dx + \int 2\,dx + \int e^{-2x}\,dx$

$= \frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + C$

$(e^x + e^{-x})^2 = (e^x)^2 + 2\cancel{e^x}\,\cancel{e^{-x}} + (e^{-x})^2$

$= e^{2x} + 2 + e^{-2x}$

Here it is convenient to observe that for any constant k,

$$\int e^{kx}\,dx \underset{\uparrow}{=} \frac{1}{k}\int e^u\,du = \frac{1}{k}e^u + C = \frac{1}{k}e^{kx} + C$$

substitute $\begin{cases} u = kx \\ du = k\,dx \end{cases}$

⑦ $$\int_0^1 \frac{\sqrt{1+e^{-x}}}{e^x}\,dx = \int_0^1 (1+e^{-x})^{1/2} e^{-x}\,dx$$

$$= -\int_2^{1+\frac{1}{e}} u^{1/2}\,du = -\frac{2}{3}u^{3/2}\Big|_{u=2}^{1+1/e}$$

$$= -\frac{2}{3}(1+1/e)^{3/2} + \frac{4\sqrt{2}}{3}$$

substitute

$u = 1+e^{-x}$

$du = -e^{-x}\,dx$

$u(0) = 1+e^0 = 2$

$u(1) = 1+\frac{1}{e}$

log and exponential functions with base $b>0$

- $\boxed{b^x = e^{x\ln(b)}}$ ← important
- $\log_b x = y \iff b^y = x, \quad x>0$
- These are inverse functions for fixed $b>0$:

$$\log_b(b^x) = x$$

$$b^{\log_b(x)} = x \quad \text{for } x>0$$

- $\frac{d}{dx}[b^x] = \ln(b)\, b^x$
- $\int b^x\, dx = \frac{1}{\ln(b)} b^x, \quad b \neq 1$
- $\log_b(x) = \ln(x)/\ln(b) \quad b \neq 1$
- $\frac{d}{dx}[\log_b(x)] = \frac{1}{x\ln(b)} \quad b \neq 1$

example

both base and exponent are variable

$$f(x) = x^x = e^{x\ln(x)} \quad , \quad x>0$$

$$f'(x) = \frac{d}{dx}\left[e^{x\ln(x)}\right] = e^{x\ln(x)} \cdot \frac{d}{dx}[x\ln x]$$

$$= x^x(\ln(x)+1)$$

side calculation:

$$\frac{d}{dx}[x\ln(x)] = \frac{d}{dx}[x]\ln x + x\frac{d}{dx}[\ln x]$$

$$= 1\cdot\ln(x) + x\frac{1}{x} = \ln(x)+1$$

Unexpected consequence of side calculation:

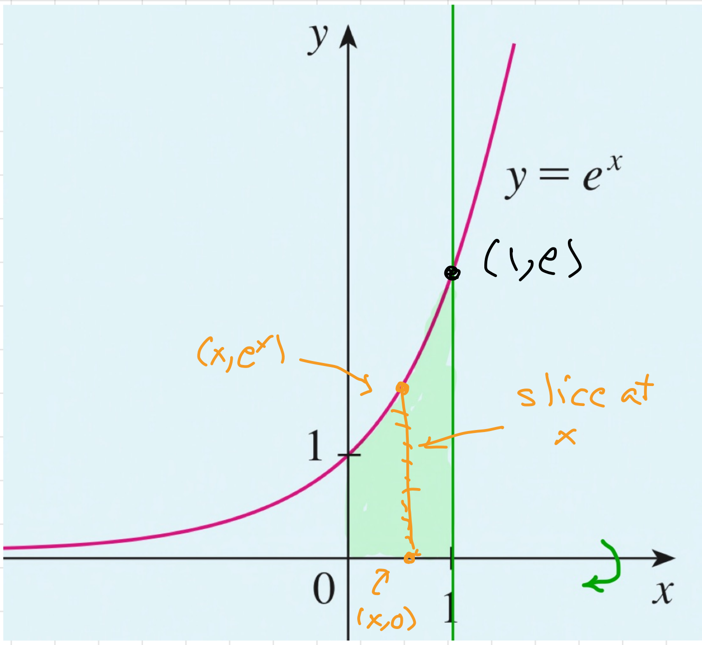
$$\int \ln(x)+1\,dx = x\ln(x) + C$$

$$\int \ln(x)dx + \int 1dx = \int \ln x\,dx + x$$

$$\int \ln(x)\,dx = x\ln(x) - x + C$$

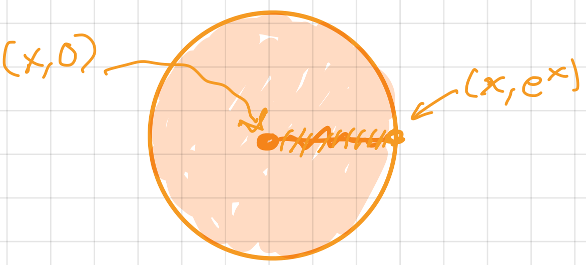
page 454:

97. Find the volume of the solid obtained by rotating about the x-axis the region bounded by the curves $y = e^x$, $y = 0$, $x = 0$, and $x = 1$.



washer method: (recommended)

For $0 \le x \le 1$, rotate the slice at x to get a disk of radius e^x, and area πe^{2x}

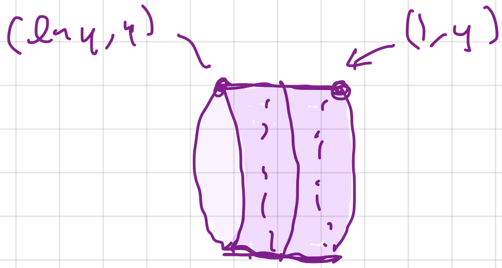


Volume of solid $= \int_0^1 \pi e^{2x} dx = \frac{\pi}{2} e^{2x} \Big|_0^1 = \frac{\pi}{2}(e^2 - 1)$

shell method:

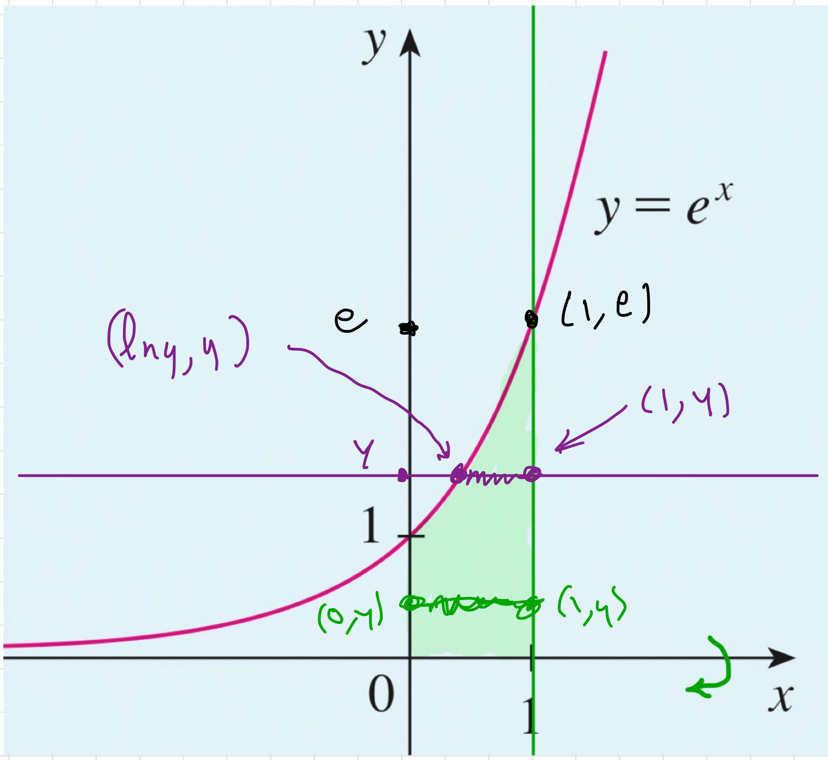
break into two pieces $1 \le y \le e$ and $0 \le y \le e$

first piece: For $1 \le y \le e$ rotate slice at y to get a shell



with width $1 - \ln(y)$, radius y and area $2\pi y(1 - \ln(y))$

second piece: For $0 \le 1 \le e$ get a shell with area $2\pi y$



Volume of solid $=$

$$\int_1^e 2\pi y(1 - \ln y)\,dy + \int_0^1 2\pi y\,dy$$