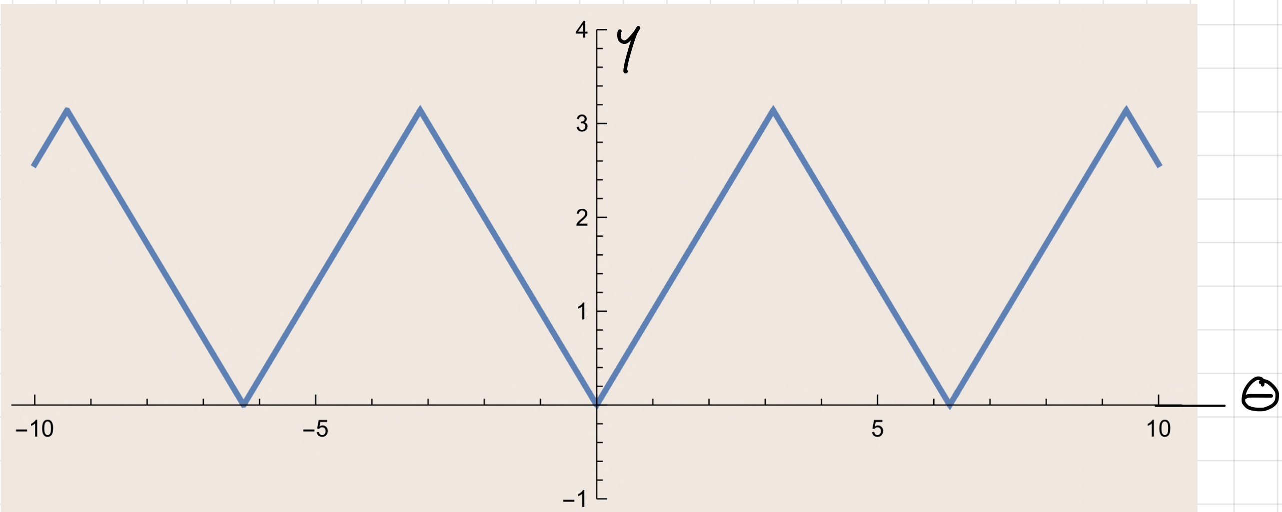
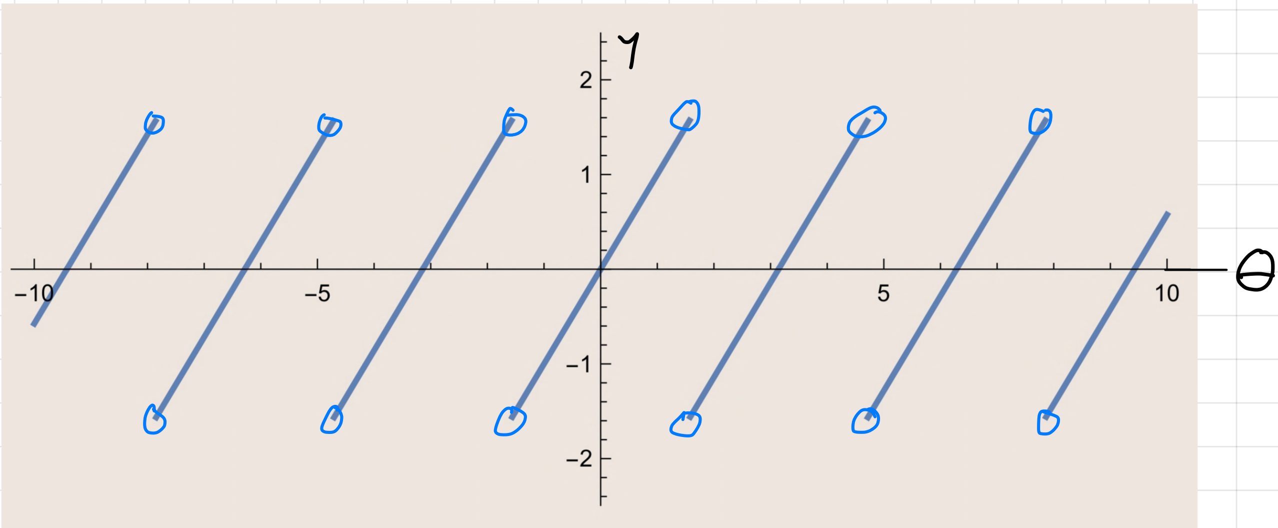
Identify the 3 graphs:

- $y = \arctan(\tan\theta)$ <u>b</u>
- $y = \arcsin(\sin\theta)$ <u>c</u>
- $y = \arccos(\cos\theta)$ <u>a</u>

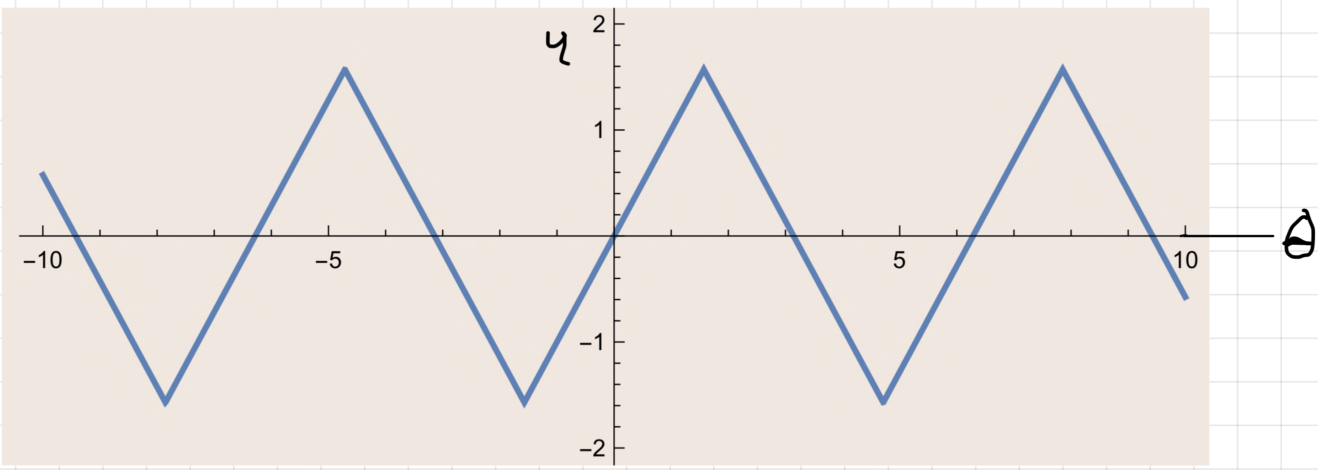
(a)



(b)



(c)



$\boxed{\text{footnote 1}}$ ← refer to last pages of this document.

Announcements

Trig Review Session:

Today 6:30 – 8:00.

Zoom link on Canvas

wwork 11 now open

due on Monday night.

$f(x) = \sin x$ $f'(x) = g(x)$

$g(x) = \cos x$ $g'(x) = -f(x)$

$f''(x) = -f(x)$ $f^{(4)}(x) = f(x)$

$g''(x) = -g(x)$ $g^{(4)}(x) = g(x)$

Already these properties tell us that sine and cosine are very important functions in calculus.

For $-1 \le x \le 1$, $\arccos(x) = \begin{cases} \text{the angle } \theta \text{ with } \cos\theta = x \\ \text{and } 0 \le \theta \le \pi \end{cases}$

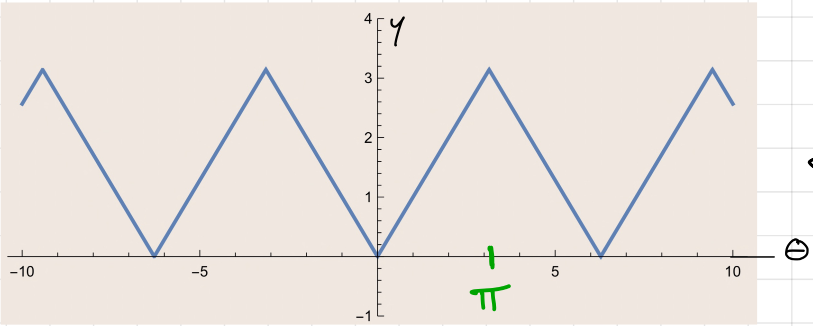
footnote 2

$\Rightarrow \arccos(\cos\theta) = \theta$ for $0 \le \theta \le \pi$

$\Rightarrow y = \arccos(\cos\theta)$ same as $y = \theta$ on $[0, \pi]$
and (a) is the only graph satisfying this

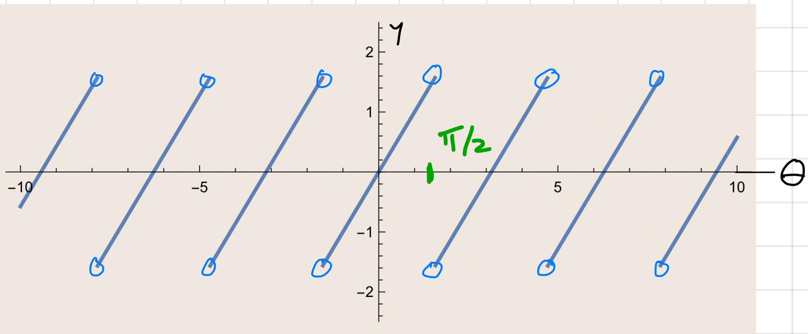
$y = \theta$ is a line with slope 1 thru origin

(a)



$\leftarrow y = \arccos(\cos\theta)$

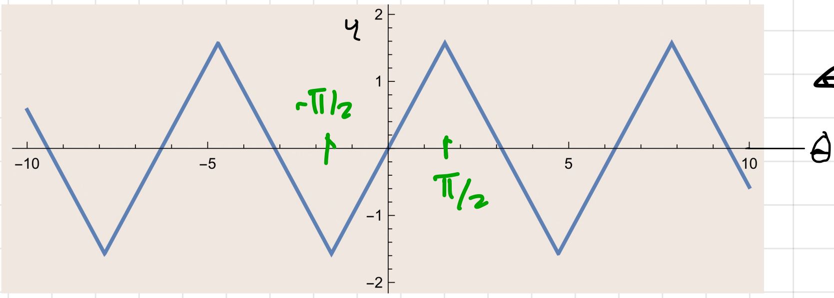
(b)



$\leftarrow y = \arctan(\tan\theta)$

$\pi/2$ is not in domain b/c $\tan(\pi/2) = \text{DNE}$

(c)



$\leftarrow y = \arcsin(\sin\theta)$

For $-1 \le x \le 1$, $\arcsin(x) = \begin{cases} \text{the angle } \theta \text{ with } \sin\theta = x \\ \text{and } -\pi/2 \le \theta \le \pi/2 \end{cases}$

$\Rightarrow y = \arcsin(\sin\theta)$ same as $y = \theta$ on $[-\pi/2, \pi/2]$

For any x, $\arctan(x) = \begin{cases} \text{the angle } \theta \text{ with } \tan\theta = x \\ \text{and } -\pi/2 < \theta < \pi/2 \end{cases}$

$\Rightarrow y = \arctan(\tan\theta)$ same as $y = \theta$ on $(-\pi/2, \pi/2)$

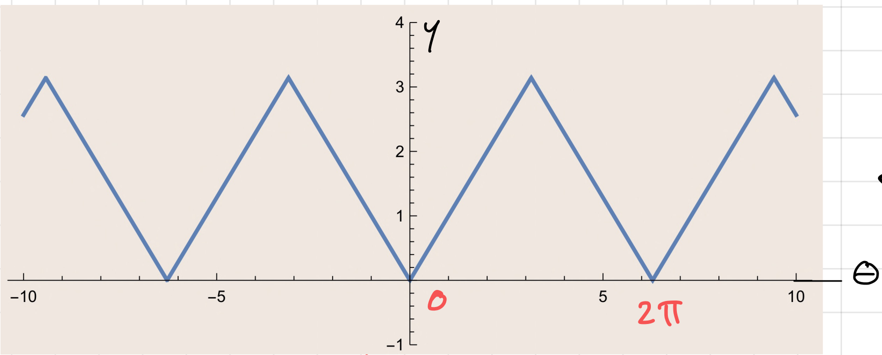
footnote 3

footnote6

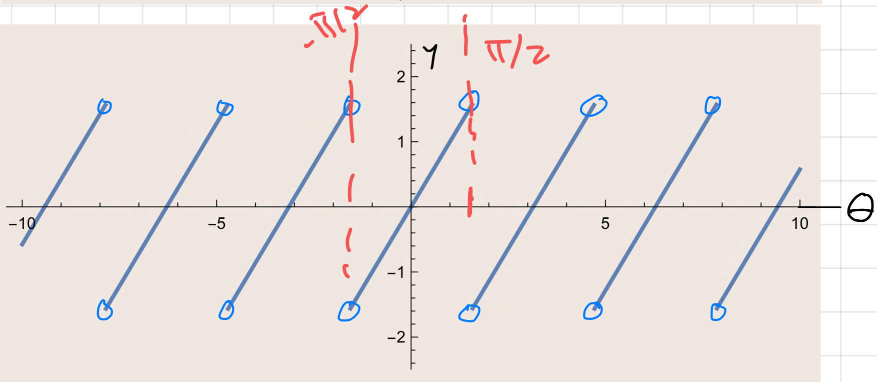
Some More Observations

- $\arccos(x)$ is always ≥ 0. So graph of $y = \arccos(\cos\theta)$ is on or above the θ-axis.
- $\arctan(\tan\theta)$ has period π footnote4
- $\arccos(\cos\theta)$ and $\arcsin(\sin\theta)$ have period 2π
- $\arccos(\cos\theta)$ is an <u>even</u> function footnote5
- $\arcsin(\sin\theta)$ and $\arctan(\tan\theta)$ are <u>odd</u>

$\arccos(\cos(-\theta)) = \arccos(\cos\theta)$

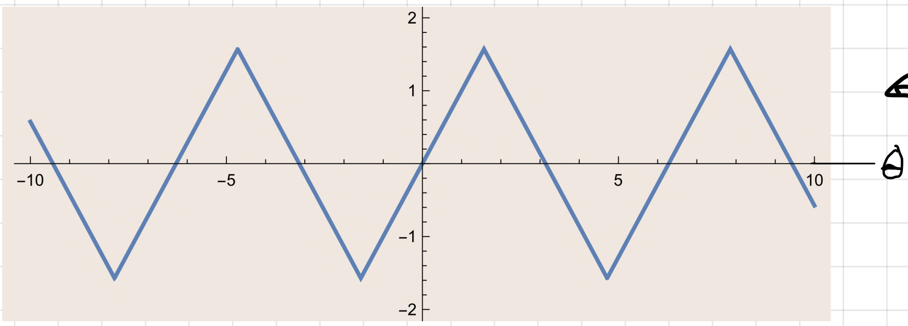


← $y = \arccos(\cos\theta)$



← $y = \arctan(\tan\theta)$

footnote7



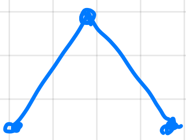
← $y = \arcsin(\sin\theta)$

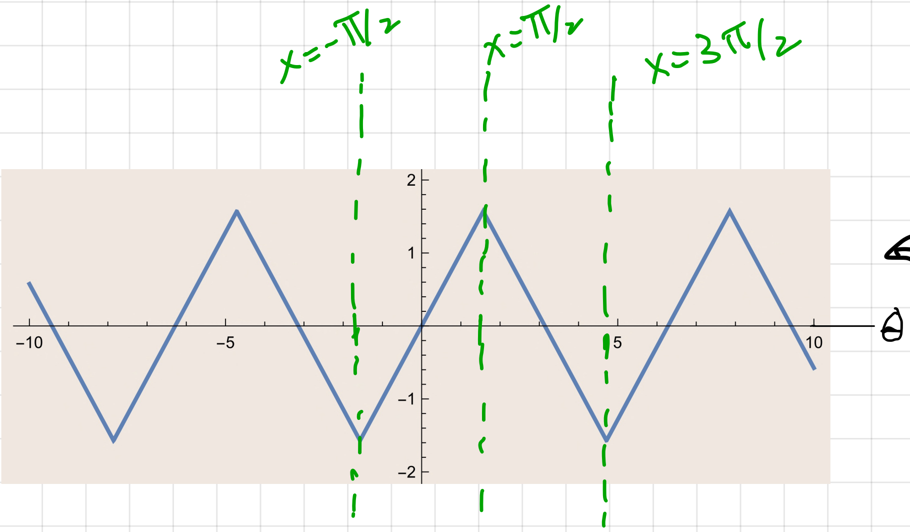
Consider $g(\theta) = \arcsin(\sin\theta)$

$$g'(\theta) = \frac{1}{\sqrt{1-\sin^2\theta}} \frac{d}{d\theta}[\sin\theta]$$

recall: for $-1<x<1$
$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{1}{\sqrt{\cos^2\theta}} \cos\theta = \frac{\cos\theta}{|\cos\theta|} = \begin{cases} 1 & \text{if } \cos\theta > 0 \\ -1 & \text{if } \cos\theta < 0 \end{cases}$$

For x between $-\pi/2$ and $\pi/2$, $\cos\theta > 0$ and $y = g(x)$ is a straight line segment with slope 1. For x between $\frac{\pi}{2}$ and $3\pi/2$, $\cos\theta < 0$ and $y = g(x)$ is a line segment with slope -1. So on the interval $[-\pi/2, 3\pi/2]$ the graph of $y = g(x)$ looks like  and extending it periodically we see the saw-tooth graph of the entire curve $y = g(x)$:



$\leftarrow y = \arcsin(\sin\theta)$

Note $g(\theta) = \arcsin(\sin\theta)$ is a continuous function with domain $\mathbb{R} = (-\infty, \infty)$

Calculus Properties of inverse trig functions

$$\frac{d}{dx}\left[\arcsin(x)\right] = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

$$\frac{d}{dx}\left[\arccos(x)\right] = \frac{-1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

$$\frac{d}{dx}\left[\arctan(x)\right] = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left[\operatorname{arcsec}(x)\right] = \frac{1}{x\sqrt{x^2-1}}, \quad |x| > 1$$

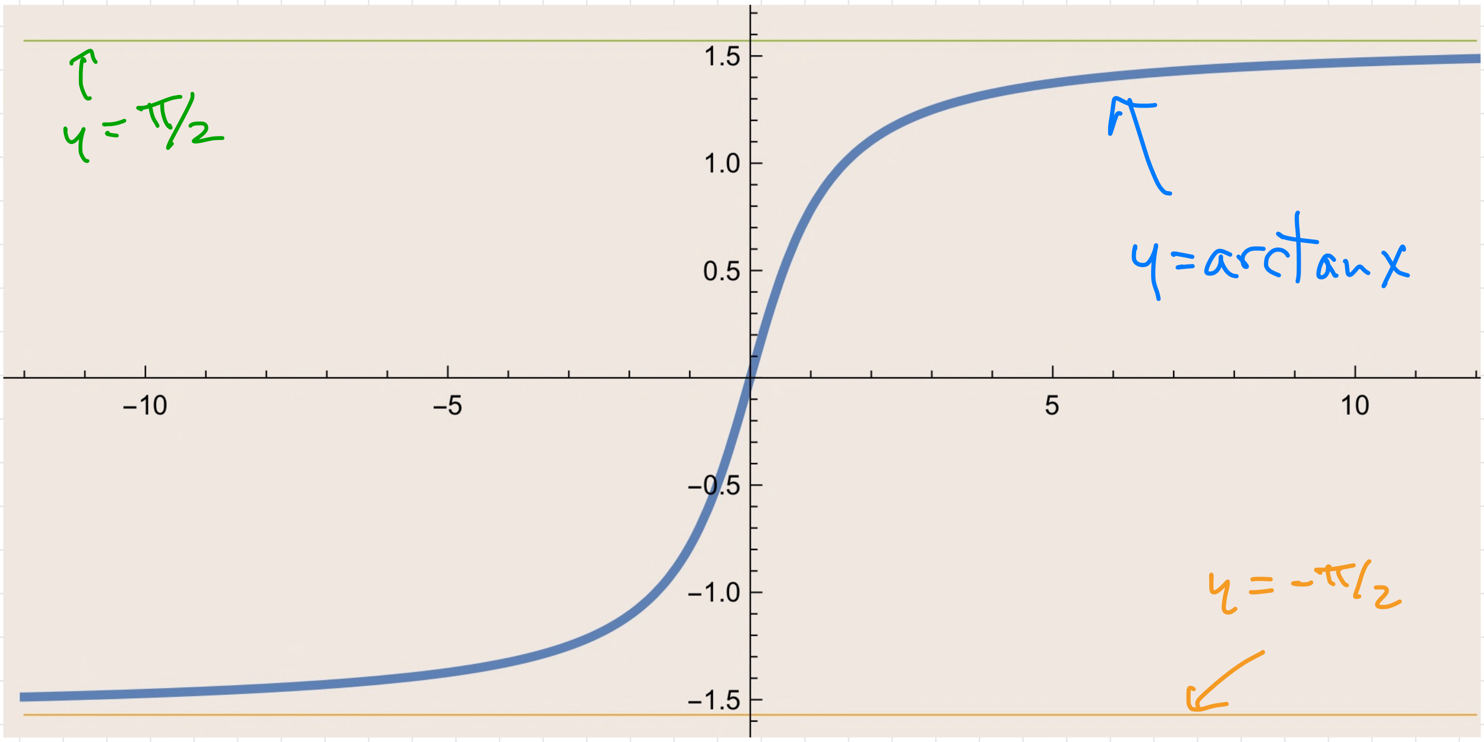
And corresponding integrals such as

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \operatorname{arcsec}(x) + C, \quad |x| > 1$$

Next we'll examine the derivative of the arctan function ...

Recall the definition:

$$\boxed{\begin{aligned} &\tan(\arctan x) = x \quad \text{for any real number } x \\ &\arctan(\tan x) = x \quad \text{for } -\pi/2 < x < \pi/2 \end{aligned}}$$



Differentiate $\tan(\arctan(x)) = x$ to get

$$1 = \frac{d}{dx}[x] = \frac{d}{dx}\left[\tan(\arctan x)\right] = \sec^2(\arctan x)\frac{d}{dx}[\arctan x]$$

$$\frac{d}{dx}[\arctan x] = \frac{1}{\sec^2(\arctan x)}$$

see next page

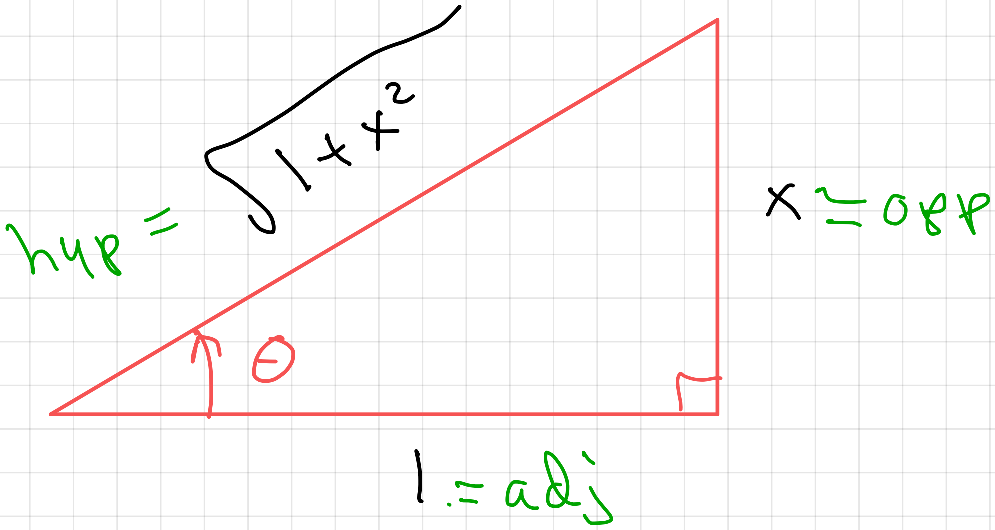
$$= \cos^2(\arctan x) = \frac{1}{1+x^2}$$

Note the two horizontal asymptotes for $y = \arctan x$

$$\lim_{x\to\infty} \arctan(x) = \pi/2$$

$$\lim_{x\to-\infty} \arctan(x) = -\pi/2$$

What does cos(arctan x) equal?



$$\theta = \arctan(x) \Rightarrow \tan\theta = \frac{x}{1} = \frac{\text{opp}}{\text{adj}}$$

$$\underset{\cos(\theta)}{\underset{\shortparallel}{\cos(\arctan x)}} = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{1+x^2}}$$

$$\cos^2(\arctan x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left\{\arctan x\right\} = \frac{1}{1+x^2} \Rightarrow \int \frac{1}{1+x^2}\,dx = \arctan x + C$$

Some Integration Problems

① $\displaystyle\int \frac{1}{7+2x^2}\,dx$

$$\frac{2}{7}x^2 = \left(\sqrt{\frac{2}{7}}\,x\right)^2$$

aside: integrand $= \dfrac{1}{7+2x^2}\,\dfrac{1/7}{1/7} = \dfrac{1}{7}\,\dfrac{1}{1+\frac{2}{7}x^2}$

$$= \frac{1}{7}\,\frac{1}{1+\left(\sqrt{2/7}\,x\right)^2} \Longrightarrow$$

$$\int \frac{1}{7+2x^2}\,dx = \frac{1}{7}\int \frac{1}{1+\left(\sqrt{2/7}\,x\right)^2}\,dx \qquad \begin{cases} u = \sqrt{\frac{2}{7}}\,x \\ du = \sqrt{\frac{2}{7}}\,dx \end{cases}$$

$$= \frac{1}{7}\sqrt{\frac{7}{2}}\int \frac{1}{1+\left(\sqrt{2/7}\,x\right)^2}\,\sqrt{\frac{2}{7}}\,dx$$

$\boxed{\leftarrow \text{mistake from class fixed}}$

$$= \frac{1}{7}\sqrt{7/2}\int \frac{1}{1+u^2}\,du = \frac{1}{7}\sqrt{7/2}\,\arctan(u) + C$$

$$= \frac{1}{\sqrt{14}}\arctan\left(\sqrt{2/7}\,x\right) + C$$

More generally, for $\displaystyle\int \frac{1}{A+Bx^2}\,dx \quad (A, B \neq 0)$

write $\dfrac{1}{A+Bx^2} = \dfrac{1}{A}\,\dfrac{1}{1+\left(\sqrt{B/A}\,x\right)^2}$

and substitute $u = \sqrt{B/A}\,x$, $du = \sqrt{B/A}\,dx$

② $$\int \frac{1}{x^2+2x+2}\,dx$$

aside

$x^2+2x+2 =$

$1+(x^2+2x+1) =$

$1+(x+1)^2$

$$\parallel$$

$$\int \frac{1}{1+(x+1)^2}\,dx$$

$$= \int \frac{1}{1+u^2}\,du \quad \leftarrow \quad \begin{cases} u = x+1 \\ du = dx \end{cases}$$

$$= \arctan(u) + C = \arctan(x+1) + C$$

③ $$\frac{1+x+x^2}{x^3+x} = \frac{(1+x^2)+x}{(1+x^2)x}$$

$$= \frac{1+x^2}{(1+x^2)x} + \frac{x}{(1+x^2)x} = \frac{1}{x} + \frac{1}{1+x^2}$$

$$\Rightarrow \int \frac{1+x+x^2}{x^3+x}\,dx = \int \frac{1}{x} + \frac{1}{1+x^2}\,dx$$

$$= \ln|x| + \arctan(x) + C$$

$$\boxed{\frac{1+x+x^2}{x^3+x} = \frac{\text{degree 2 polynomial}}{\text{degree 3 polynomial}} = \text{A rational function}}$$

These examples show that ln(x) and arctan(x) can be very useful for integrating rational functions R(x). We will expand on this in Chapter 7. (Technique of "partial fractions".)

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

$$\int \frac{1}{x^p}\,dx = \frac{x^{1-p}}{(1-p)} + C, \quad p \neq 1$$

Footnote Comments

1. The functions arcsin(sinθ), arccos(cosθ) and are not particularly important but examining them will allow us to review definitions and bring out some properties of the inverse trig functions.

2. $$\boxed{\text{For } -1 \le x \le 1,\ \text{arcos}(x) = \begin{cases} \text{the angle } \theta \text{ with } \cos\theta = x \\ \text{and } \ 0 \le \theta \le \pi \end{cases}}$$

This is the definition of arcos(x) which is the inverse function of $f(x) = \cos(x)$, $0 \le x \le \pi$.

The graph of $y = \cos(x)$ for $0 \le x \le \pi$ satisfies HLP so we know that $f(x)$ has an inverse function. We have also summarized this definition by writing

$$\boxed{\begin{array}{ll} \cos(\arccos x) = x & \text{for } -1 \le x \le 1 \\ \arccos(\cos x) = x & \text{for } 0 \le x \le \pi \end{array}}$$

3. Since the problem posed on page 1 is a multiple choice question we can assume that the graphs of the 3 functions coincide with the 3 pictures that are drawn. So, on first pass, we can use a process of elimination to determine the answer.

4. A function $f(x)$ is <u>periodic</u> if there is a fixed number $p>0$ so that $f(x+p)=f(x)$ for all x in domain(f). The smallest $p>0$ with $f(x+p)=f(x)$ is called the <u>period</u> of $f(x)$. Of the 6 trig functions, $\tan(x)$ and $\cot(x)$ have period π but the other 4 have period 2π.

5. A function $f(x)$ is <u>even</u> if $f(-x)=f(x)$ for all x in domain(f) and <u>odd</u> if $f(-x)=-f(x)$. Of the trig functions, $\cos(x)$ and $\sec(x)$ are even, while $\sin(x)$, $\tan(x)$, $\cot(x)$ and $\csc(x)$ are odd.

6. We've solved the multiple choice problem but to really understand the 3 graphs we should dig deeper.

7. Since $\arctan(\tan\theta)$ has period π and $\arctan(\tan\theta)=\theta$ whenever θ is the interval $(-\pi/2, \pi/2)$, we can now be certain that graph (b) is the graph of $y=\arctan(\tan\theta)$.