Integration By Parts:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

example: $\int x\cos(x)\,dx = ?$

$$\begin{cases} u = x & du = \frac{du}{dx}dx = dx \\ dv = \cos x\,dx & v = \int dv = \int \cos x\,dx = \sin x \end{cases}$$

$$\int x\cos x\,dx = \int u\,dv = u\cdot v - \int v\,du$$

$$= x\cdot\sin(x) - \int \sin x\,dx$$

$$= x\cdot\sin(x) + \cos(x) + C$$

This is a prototype example for the use of IBP. For this technique to be succesful there are two key things that must happen:

- need to be able to work $\int dv$
- need to be able to work $\int v\,du$.

Keep these in mind when choosing u and dv.

$$\boxed{\int u\,dv = uv - \int v\,du}$$

example $\int x \ln(x)\,dx$

Try $\begin{cases} u = x & du = dx \\ dv = \ln x\,dx & v = \int \ln x\,dx = x\ln(x) - x \end{cases}$

$$\int u\,dv = \int x(x\ln x - x)\,dx$$
$$= \int x^2 \ln x - x^2\,dx$$

not working!

Try $\begin{cases} u = \ln(x) & du = \frac{1}{x}\,dx \\ dv = x\,dx & v = \int x\,dx = \frac{1}{2}x^2 \end{cases}$

okay!

$$\int u\,dv = u \cdot v - \int v\,du = \frac{1}{2}x^2\ln(x) - \int \frac{1}{2}x^2 \frac{1}{x}\,dx$$
$$= \frac{1}{2}x^2 \ln(x) - \frac{1}{2}\int x\,dx$$
$$= \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C$$

key things to look for:

- need to be able to work $\int dv$ ✓
- need to be able to work $\int u\,dv$. ✓

Problem: Calculate $\int \underbrace{\arctan(x)}_{u} \, \underbrace{dx}_{dv}$.

Use integration by parts:

$$\begin{cases} u = \arctan(x) \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = \frac{1}{1+x^2}\, dx \\ v = \int dx = x \end{cases}$$

Then

$$\int \arctan(x)\, dx = \int u\, dv = uv - \int v\, du$$

$$= x\arctan(x) - \frac{1}{2}\int 2x \frac{1}{1+x^2}\, dx$$

$w > 0$

$$= x\arctan(x) - \frac{1}{2}\int \frac{1}{w}\, dw \qquad \begin{cases} w = 1+x^2 \\ dw = 2x\, dx \end{cases}$$

$$= x\arctan(x) - \frac{1}{2}\ln|w| + C$$

$$= x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C$$

(this is integral formula 89 in back of Stewart)

$$= x\arctan(x) - \ln\left(\sqrt{1+x^2}\right) + C$$

Look at discussion and problems in <u>Section 7.5</u> in Stewart

Integrating Trig Functions

$\int \sin x \, dx = -\cos x + C$

$\int \cos x \, dx = \sin x + C$

$\int \frac{\sin x}{\cos x} \, dx$

$\int \tan x \, dx = \ln|\sec x| + C = -\ln|\cos x| + C$

$\int \sec x \, dx = \ln|\sec x + \tan x| + C$ (see next page)

$\int \cot x \, dx = \ln|\sin x| + C$

$\int \csc x \, dx = \ln|\csc x - \cot x| + C$

$\int \sec^2 x \, dx = \tan x + C$

$\int \sec x \tan x \, dx = \sec x + C$

- $\int \sec x \, dx$

aside:

$$\sec x = \sec x \, \frac{\sec x + \tan x}{\sec x + \tan x}$$

$$= \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

substitute

$u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x) \, dx$

$$= \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C$$

(formula #14 in back of Stewart)

What about products and powers of trig functions?

Examples

$\begin{cases} u = \sin x \\ du = \cos x \, dx \end{cases}$

① $\int \underbrace{\sin x}_{u} \underbrace{\cos x \, dx}_{du} = \int u \, du = \frac{u^2}{2} + C = \frac{1}{2}\sin^2 x + C$

This integral is very important!

② $\int \cos^2(x) \, dx$

$= \int \frac{1}{2}(1 + \cos 2x) \, dx$

see next page

$\boxed{\cos^2(x) = \frac{1}{2}(1 + \cos 2x)}$

$= \frac{x}{2} + \frac{1}{2} \cdot \frac{1}{2} \sin(2x) + C$

$= \frac{x}{2} + \frac{1}{4} \sin 2x + C$

$\int \cos 2x \, dx$

$= \frac{1}{2} \sin(2x) + C$

(Formula #64 in back of Stewart.)

③ $\int \sin^2(x) \, dx = \int 1 - \cos^2(x) \, dx$

$= x - \left(\frac{x}{2} + \frac{1}{4} \sin 2x\right) + C = \frac{x}{2} - \frac{1}{4} \sin 2x + C$

(Formula #63 in back of Stewart.)

The "half-angle formula" is one of many special cases of the trig addition formula:

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B).$$

Taking $A = B = x$ gives:

$$\begin{aligned} \cos(2x) &= \cos^2(x) - \sin^2(x) \\ &= \cos^2(x) - (1 - \cos^2(x)) \\ &= 2\cos^2(x) - 1 \end{aligned}$$

and solving this equation for $\cos^2(x)$ gives

$$\cos^2(x) = \frac{1}{2}(\cos(2x) + 1)$$