Horizontal Asymptotes for $y = f(x)$

If $\lim_{x\to\infty} f(x) = L$ (where $L \neq \pm\infty$) then the graph of $y=f(x)$ has $y = L$ as an asymptote on the right.

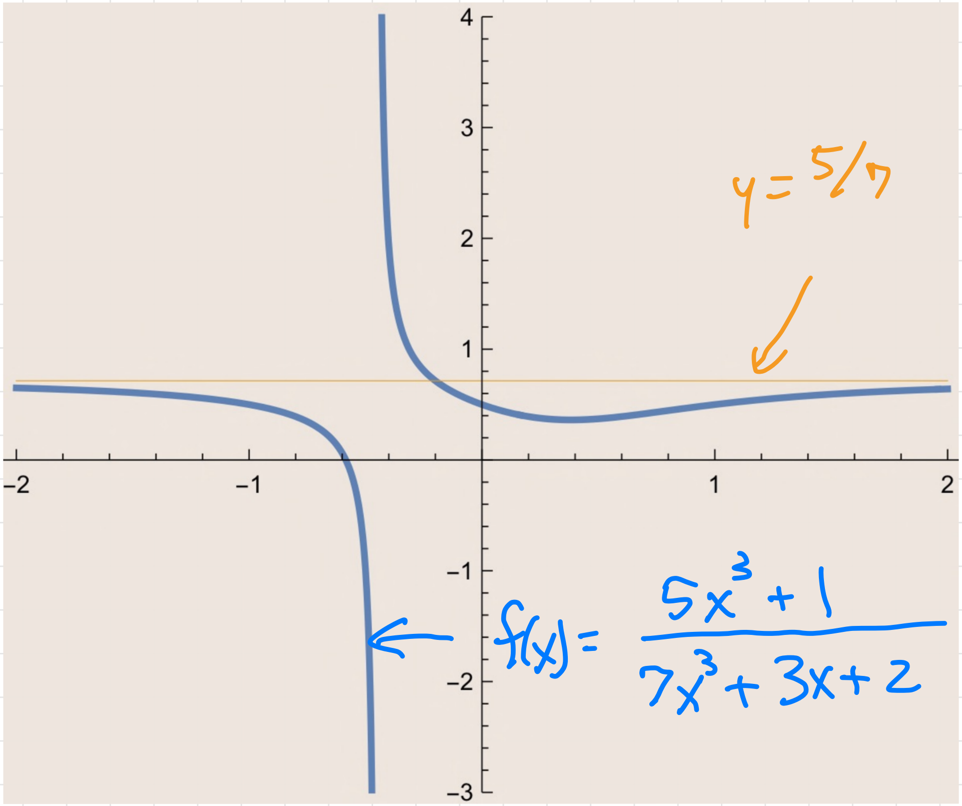
If $\lim_{x\to-\infty} f(x) = L$ (where $L \neq \pm\infty$) then the graph of $y=f(x)$ has $y = L$ as an asymptote on the left.

Example

$$f(x) = \frac{5x^3+1}{7x^3+3x+2} = \frac{5+\underbrace{1/x^3}_{\to 0}}{7+\underbrace{3/x^2}_{\to 0}+\underbrace{2/x^3}_{\to 0}}$$

$\Rightarrow$

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 5/7 \Rightarrow$$ $y = 5/7$ is asymptote on the left and the right.

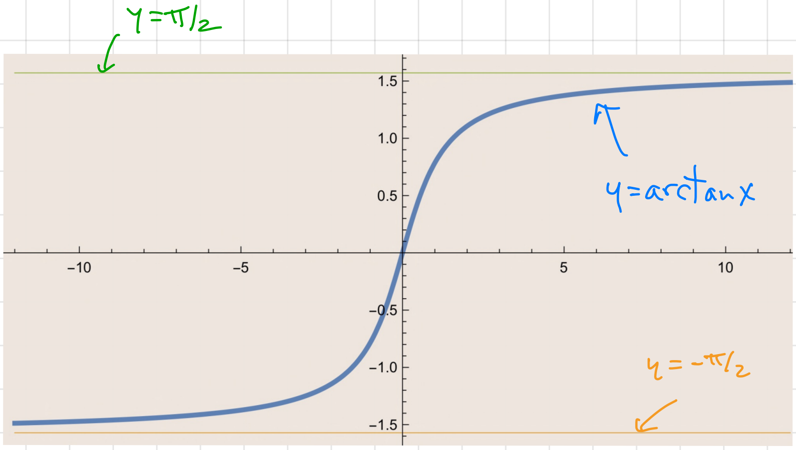


In fact, for any rational function $f(x) = P(x)/Q(x)$, if $y = L$ is a horizontal asymptote for $y = f(x)$ on the right then it is also a horizontal asymptote on the left.

But this is not true for more general functions. $\rightarrow$

Horizontal Asymptotes (continued)

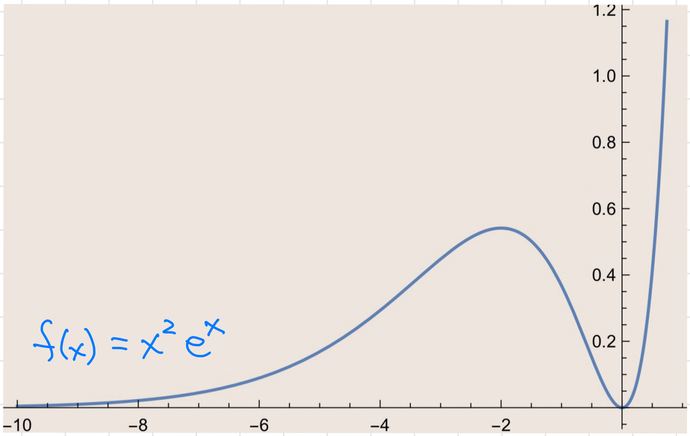
(1)



different horiz. asym. on right and left.

$$\lim_{x\to\infty} \tan^{-1}(x) = \pi/2, \quad \lim_{x\to-\infty} \tan^{-1}(x) = -\pi/2$$

(2)



no horiz. asym. on right.

$$\lim_{x\to-\infty} x^2 e^x = 0, \quad \lim_{x\to\infty} x^2 e^x = \infty$$

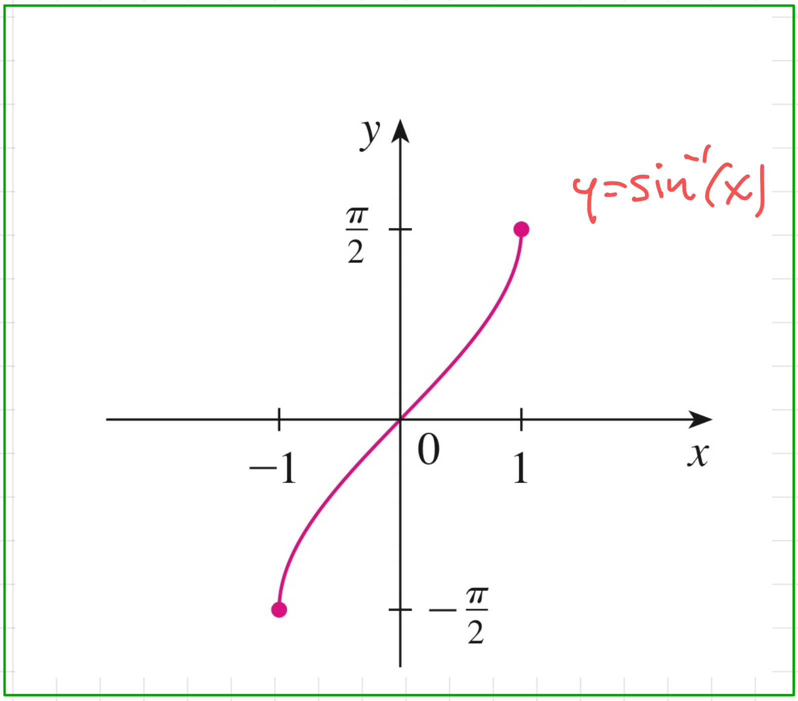
(3)

$$\lim_{x\to\infty} \sin^{-1}(x) = DNE$$

$$\lim_{x\to-\infty} \sin^{-1}(x) = DNE$$

domain$(\sin^{-1}) = [-1,1]$

No horiz. asymptotes at all.



u times dv

Integration by Parts (IBP):

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Each of the following integrals can be worked using IBP. How should u be chosen?

	u	dv
① $\int x^3 \cos(2x)\,dx$	x^3 $du = 3x^2\,dx$	$\cos(2x)\,dx$ $v = \frac{1}{2}\sin(2x)$
② $\int \arcsin(x)\,dx$	$u = \arcsin x$ $du = \frac{1}{\sqrt{1-x^2}}\,dx$	$dv = dx$ $v = x$
③ $\int \cos(x)\,e^x\,dx$	$u = \cos(x)$ $du = -\sin(x)\,dx$	$dv = e^x\,dx$ $v = e^x$
④ $\int \sec^3(x)\,dx$	$u = \sec(x)$ $du = \sec(x)\tan(x)\,dx$	$dv = \sec^2(x)\,dx$ $v = \tan(x)$

① $\int x^3 \cos(2x)\,dx$ $\quad \begin{cases} u = x^3 & du = 3x^2\,dx \\ dv = \cos(2x)\,dx & v = \frac{1}{2}\sin(2x) \end{cases}$

$$= \underbrace{\frac{x^3}{2}\sin(2x)}_{u\cdot v} - \int \underbrace{\frac{3}{2}x^2\sin(2x)\,dx}_{v\,d}$$

$= \cdots$ repeat twice more $\cdots$ (IBP)

② $\int \arcsin(x)\,dx$ $\quad \begin{cases} u = \sin^{-1}(x) & du = \frac{1}{\sqrt{1-x^2}}\,dx \\ dv = dx & v = x \end{cases}$

$$= \underbrace{x\sin^{-1}(x)}_{u\cdot v} - \int \overbrace{\frac{x}{\sqrt{1-x^2}}\,dx}^{v\,du}$$

$= \cdots$ substitute $w = 1-x^2$, $dw = -2x\,dx$

$$= x\sin^{-1}(x) + \sqrt{1-x^2} + C$$

Stewart Table of Integrals #87

(3) $\int \cos(x)\, e^x \, dx$ $\begin{cases} u = e^x & du = e^x dx \\ dv = \cos x \, dx & v = \sin x \end{cases}$

$$= \underbrace{e^x \sin(x)}_{uv} - \underbrace{\int e^x \sin(x) dx}_{\int v\, du}$$

$\begin{cases} u_1 = e^x & du_1 = e^x dx \\ dv_1 = \sin(x)\, dx & v_1 = -\cos(x) \end{cases}$

$$= e^x \sin(x) - \left(\underbrace{-e^x \cos x}_{u_1 v_1} - \underbrace{\int -\cos(x)\, e^x \, dx}_{\int v_1 du_1} \right)$$

$$= e^x \sin(x) + e^x \cos(x) - \int \cos(x)\, e^x \, dx$$

We have shown

$$2\int \cos(x) e^x dx = e^x(\sin(x) + \cos(x)) \cancel{- \int \cos(x)\, e^x \, dx}$$

$\Rightarrow$

$$\int \cos(x) e^x dx = \frac{e^x}{2}(\sin(x) + \cos(x)) + C$$

$$\boxed{\tan^2(x)+1=\sec^2(x)}$$

④ $2\int \sec^3(x)\,dx$

$$\begin{cases} u=\sec(x) & du=\sec(x)\tan(x)\,dx \\ dv=\sec^2(x)\,dx & v=\tan(x) \end{cases}$$

$$=\underbrace{\sec(x)\tan(x)}_{u\cdot v}-\underbrace{\int \tan(x)\sec(x)\tan(x)\,dx}_{\int v\,du}$$

$$=\sec(x)\tan(x)-\int \tan^2(x)\sec(x)\,dx$$

$$=\sec(x)\tan(x)-\int (\sec^2(x)-1)\sec(x)\,dx$$

$$=\sec(x)\tan(x)+\int \sec(x)\,dx-\cancel{\int \sec^3(x)\,dx}$$

<u>recall</u> $\int \sec(x)\,dx=\ln|\sec(x)+\tan(x)|+C$

Conclude by trick similar to ③ that:

$$\int \sec^3 x\,dx=\frac{1}{2}\sec x\tan x+\frac{1}{2}\ln|\sec x+\tan x|+C$$

<u>Stewart Formula 71</u>

Announcements

- Exam 3 on Monday
- Problem Review Session, Saturday morning
- wwork 13 due Saturday 11:59 pm

Integrating Trig Functions

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \tan x \, dx = \ln|\sec x| + C = -\ln|\cos x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \cot x \, dx = \ln|\sin x| + C$$

$$\int \csc x \, dx = \ln|\csc x - \cot x| + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

What about products and powers of trig functions?

There a few trig identities that are very useful for integration:

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\sin(2x) = 2\sin x \cos x$$

$$\cos(2x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

These four are consequences of the addition formulas for sine and cosine.

use $\frac{1}{2}$ angle formula

<u>example</u>

$$\int \sin^2(x)\,dx = \int \frac{1}{2} - \frac{1}{2}\cos(2x)\,dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

$$= \frac{1}{2}x - \frac{1}{2}\sin(x)\cos(x) + C$$

<u>Formula 63 in Stewart's Table</u>

<u>example</u> $\int \sin^4(\theta)\cos^3(\theta)\,d\theta$

$$= \int \sin^4(\theta)\cos^2(\theta)\cos(\theta)\,d\theta$$

$$= \int \sin^4(\theta)(1-\sin^2(\theta))\cos(\theta)\,d\theta$$

$$= \int u^4(1-u^2)\,du \qquad \left(\begin{array}{l} u = \sin\theta \\ du = \cos\theta\,d\theta \end{array}\right.$$

$$= \int u^4 - u^6\,du$$

$$= \frac{1}{5}u^5 - \frac{1}{7}u^7 + C = \frac{1}{5}\sin^5\theta - \frac{1}{7}\sin^7\theta + C$$

Section 7.5 Stewart In this section Stewart talks about strategies to decide what technique of integration to use on which integrals. The exercises at the end of this section are very good ones to work on to build up your expertise for calculating integrals. However some of these problems use techniques from sections 7.3 and 7.4 that we haven't yet discussed (called 'trig substitution' and 'partial fractions').

Below I have circled problems in purple that can be worked with current knowledge. And for some problems hints are given to get started. Working through these problems using the hints should really help to master the topics in Chapter 7.

Reminder: "Integration is hard"

7.5 EXERCISES

1–82 Evaluate the integral.

1. $\displaystyle\int \frac{\cos x}{1 - \sin x}\,dx$

2. $\displaystyle\int_0^1 (3x + 1)^{\sqrt{2}}\,dx$

3. $\displaystyle\int_1^4 \sqrt{y}\,\ln y\,dy$

4. $\displaystyle\int \frac{\sin^3 x}{\cos x}\,dx$

5. $\displaystyle\int \frac{t}{t^4 + 2}\,dt$

~~**6.**~~ $\displaystyle\int_0^1 \frac{x}{(2x + 1)^3}\,dx$

7. $\displaystyle\int_{-1}^1 \frac{e^{\arctan y}}{1 + y^2}\,dy$

8. $\displaystyle\int t \sin t \cos t\,dt$

~~**9.**~~ $\displaystyle\int_2^4 \frac{x + 2}{x^2 + 3x - 4}\,dx$

10. $\displaystyle\int \frac{\cos(1/x)}{x^3}\,dx$

~~**11.**~~ $\displaystyle\int \frac{1}{x^3\sqrt{x^2 - 1}}\,dx$

~~**12.**~~ $\displaystyle\int \frac{2x - 3}{x^3 + 3x}\,dx$

13. $\displaystyle\int \sin^5 t\,\cos^4 t\,dt$

~~**14.**~~ $\displaystyle\int \ln(1 + x^2)\,dx$

15. $\displaystyle\int x \sec x \tan x\,dx$

~~**16.**~~ $\displaystyle\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1 - x^2}}\,dx$

17. $\displaystyle\int_0^{\pi} t \cos^2 t\,dt$

18. $\displaystyle\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt$

19. $\displaystyle\int e^{x+e^x}\,dx$

20. $\displaystyle\int e^2\,dx$

~~**21.**~~ $\displaystyle\int \arctan\sqrt{x}\,dx$

22. $\displaystyle\int \frac{\ln x}{x\sqrt{1 + (\ln x)^2}}\,dx$

Some Hints

#1: multiply by $(1+\sin x)/(1+\sin x)$. (see next page)

#2: $\int x^p\,dx = \frac{1}{p+1}x^{p+1} + C$. Take $p = \sqrt{2}$.

#3: Try IBP with $u = \ln(y)$

#4: $\dfrac{\sin^3(x)}{\cos(x)} = \dfrac{1-\cos^2(x)}{\cos(x)} \cdot \sin(x)$

#5: substitute $u = t^2$ and observe $t^4 = u^2$

#7: $\dfrac{d}{dy}\left[\arctan(y)\right] = ?$

#8: IBP

#10: $u = 1/x$ (see next page)

#13: $\sin^5 t = (1-\cos^2 t)^2 \sin t$

#17: Try IBP with $u = t\cos t$

#19: $e^{x+e^x} = e^{e^x} e^x$ take $u = e^x$

#20: easiest of all

#22: substitute $u = 1 + \ln(x)^2$

#1: $\dfrac{\cos(x)}{1-\sin(x)} \cdot \dfrac{1+\sin(x)}{1+\sin(x)} = \dfrac{\cos(x)+\sin(x)\cos(x)}{1-\sin^2(x)}$

$= \dfrac{\cos(x)+\sin(x)\cos(x)}{\cos^2(x)}$

$= \dfrac{\cos(x)}{\cos^2(x)} + \dfrac{\sin(x)\cos(x)}{\cos^2(x)}$

$= \dfrac{1}{\cos(x)} + \dfrac{\sin(x)}{\cos(x)} = \sec(x) + \tan x$

So $\displaystyle\int \frac{\cos x}{1-\sin x}\,dx = \int \sec(x) + \tan(x)\,dx =$

$= \ln|\sec x + \tan x| - \ln|\cos x| + C$

#10: Substitute $u = 1/x$, $du = -\frac{1}{x^2}\,dx$

and observe that $\frac{1}{x^3}\,dx = \left(-\frac{1}{x}\right)\left(-\frac{1}{x^2}\,dx\right) = -u\,du$.

Then $\displaystyle\int \frac{\cos(1/x)}{x^3}\,dx = -\int u\cos(u)\,du$ now use IBP.

#31: $\sqrt{\dfrac{1+x}{1-x}} = \sqrt{\dfrac{1+x}{1-x} \cdot \dfrac{1+x}{1+x}}$

$= \sqrt{\dfrac{(1+x)^2}{1-x^2}} = \dfrac{1+x}{\sqrt{1-x^2}}$

substitute $\begin{cases} u = 1-x^2 \\ du = -2x\,dx \end{cases}$

So $\displaystyle\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\,dx$

$= \sin^{-1}(x) - \sqrt{1-x^2} + C$

23. $\int_0^1 (1+\sqrt{x})^8\,dx$

24. $\int (1+\tan x)^2 \sec x\,dx$

25. $\int_0^1 \frac{1+12t}{1+3t}\,dt$

26. $\int_0^1 \frac{3x^2+1}{x^3+x^2+x+1}\,dx$

27. $\int \frac{dx}{1+e^x}$

28. $\int \sin\sqrt{at}\,dt$

29. $\int \ln\left(x+\sqrt{x^2-1}\right)dx$

30. $\int_{-1}^2 |e^x-1|\,dx$

31. $\int \sqrt{\frac{1+x}{1-x}}\,dx$

32. $\int_1^3 \frac{e^{3/x}}{x^2}\,dx$

33. $\int \sqrt{3-2x-x^2}\,dx$

34. $\int_{\pi/4}^{\pi/2} \frac{1+4\cot x}{4-\cot x}\,dx$

35. $\int_{-\pi/2}^{\pi/2} \frac{x}{1+\cos^2 x}\,dx$

36. $\int \frac{1+\sin x}{1+\cos x}\,dx$

37. $\int_0^{\pi/4} \tan^3\theta\,\sec^2\theta\,d\theta$

38. $\int_{\pi/6}^{\pi/3} \frac{\sin\theta\,\cot\theta}{\sec\theta}\,d\theta$

39. $\int \frac{\sec\theta\,\tan\theta}{\sec^2\theta-\sec\theta}\,d\theta$

40. $\int_0^{\pi} \sin 6x\cos 3x\,dx$

41. $\int \theta\tan^2\theta\,d\theta$

42. $\int \frac{\tan^{-1}x}{x^2}\,dx$

43. $\int \frac{\sqrt{x}}{1+x^3}\,dx$

44. $\int \sqrt{1+e^x}\,dx$

45. $\int x^5 e^{-x^3}\,dx$

46. $\int \frac{(x-1)e^x}{x^2}\,dx$

47. $\int x^3(x-1)^{-4}\,dx$

48. $\int_0^1 x\sqrt{2-\sqrt{1-x^2}}\,dx$

Hints

#24: $(1+\tan x)^2 = 1+2\tan x+\tan^2 x$
$= 2\tan x + \sec^2 x$

#30: $|e^x-1| = \begin{cases} e^x-1 & \text{if } x\geq 1 \\ 1-e^x & \text{if } x<1 \end{cases}$

#31: multiply inside square root by $\frac{1+x}{1+x}$
(see previous page)

#34: $\frac{1+4\cot x}{4-\cot x} = \frac{1+4\frac{\cos x}{\sin x}}{4-\frac{\cos x}{\sin x}} = \frac{\sin x+4\cos x}{4\sin x-\cos x}$
what's derivative of denominator?

#36: multiply $\frac{1-\cos x}{1-\cos x}$

#37 $\tan^2\theta\sec\theta = \sec^3\theta - \sec\theta$

#38,39: rewrite in terms of sin and cos

#40: $\sin(6x) = 2\sin(3x)\cos(3x)$

#45: substitute $u=-x^3$ and replace x^3 with $-u$.

Remark #35 The integrand $f(x) = \frac{x}{1+\cos^2 x}$ is an odd function, so $\int_{-a}^{a} f(x)\,dx = 0$ for any number a.